

On universal norms and the first layers of \mathbb{Z}_p -extensions of a number field

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Abstract

For an odd prime p and a number field F containing a primitive p -th root of unity, we describe the Kummer radical \mathcal{A}_F of the first layers of all the \mathbb{Z}_p -extensions of F in terms of universal norms of p -units along the cyclotomic tower of F . We also study "twisted" radicals related to \mathcal{A}_F .

Introduction

Let p be an odd prime number and F be a number field containing a primitive p -th root of unity ζ_p . We denote by \mathcal{A}_F the Kummer radical of the first layers of all the \mathbb{Z}_p -extensions of F . Precisely \mathcal{A}_F is the subgroup of $F^\bullet/F^{\bullet p}$ consisting of classes $a \bmod F^{\bullet p}$ such that the Kummer extension $F(\sqrt[p]{a})$ is contained in a \mathbb{Z}_p -extension of F . The determination of the group \mathcal{A}_F is an old problem which dates back to the beginnings of Iwasawa theory and has since been tackled by many authors. Here we present what we hope to be a satisfactory Iwasawa theoretical solution. In order to give meaning to this assertion, it is appropriate to recall that most of the (numerous) results obtained so far bring the study of \mathcal{A}_F back to that of its orthogonal complement for the Kummer pairing, namely $\mathrm{tor}_{\mathbb{Z}_p} \mathcal{X}_F/p$, where \mathcal{X}_F denotes the Galois group of the maximal abelian pro- p -extension of F which is unramified outside p -adic primes. Without pretending to be exhaustive, let us cite the following articles: [CK76], which uses the idelic description of \mathcal{X}_F to compute $\mathrm{tor}_{\mathbb{Z}_p} \mathcal{X}$ for certain quadratic fields; [Gra85], which constructs from Artin symbols a "logarithm" map on \mathcal{X}_F whose kernel is precisely the \mathbb{Z}_p -torsion $\mathrm{tor}_{\mathbb{Z}_p} \mathcal{X}_F$; [He88, Th93], which approach \mathcal{A}_F by a "dévissage" of $\mathrm{tor}_{\mathbb{Z}_p} \mathcal{X}_F$ in a local-global perspective \dots . From a cohomological point of view, and under Leopoldt's conjecture, $\mathrm{tor}_{\mathbb{Z}_p} \mathcal{X}_F$ is a "twisted dual" (see for example [Ng86] and the references therein) of the p -primary part of the tame kernel of K_2F and a question raised by Coates [Co73] was whether \mathcal{A}_F coincides with the Tate kernel of F , i.e. the subgroup \mathcal{T}_F of $F^\bullet/F^{\bullet p}$ consisting of classes $a \bmod F^{\bullet p}$ such that the symbol $\{\zeta_p, a\} = 0$ in K_2F . In this direction and still without claim of exhaustiveness, let us cite the following articles: [KC78], which establishes a "wrong duality" between the elements of order p in \mathcal{X}_F and the quotient mod p of the tame kernel R_2F ; [Gre78], which performs Iwasawa descent on the "twisted duals" of the free part of the Galois group \mathcal{X}_{F_∞} , where F_∞ is the cyclotomic \mathbb{Z}_p -extension of F .

All these approaches, by class field theory or by K -theory, produce effective methods allowing to compute \mathcal{A}_F from arithmetical parameters attached to F , such as the class group or the group of units of the field. But these descriptions of \mathcal{A}_F can not be considered as theoretically complete when they lead to other arithmetical objects such as $\mathrm{tor}_{\mathbb{Z}_p} \mathcal{X}_F$ or \mathcal{T}_F which are not necessarily better known than \mathcal{A}_F itself. Thus $\mathrm{tor}_{\mathbb{Z}_p} \mathcal{X}_F$ is, by Iwasawa theory, linked to the p -adic L -functions and its

"local-global dévissage" [He88, Ng86, Th93] leads to an Iwasawa module of descent closely related to the class group. In the same way, the "local-global dévissage" of the Tate kernel \mathcal{T}_F eventually leads to the wild kernel, which is in turn isomorphic to a twisted version of the preceding Iwasawa module of descent. In addition, the intervention of an Iwasawa descent suggests that a satisfactory theoretical description of the Kummer radical \mathcal{A}_F should include in some way an asymptotical ingredient. This is confirmed by Greenberg's answer to the question of Coates [Gre78, page 1242] : though on the ground level \mathcal{A}_F and \mathcal{T}_F are not the same in general, they coincide when going sufficiently up the cyclotomic tower. In short, a satisfactory (in our sense) description of \mathcal{A}_F should take into account the following two remarks:

- the parameters which intervene should be arithmetically or at least effectively accessible
- the answer should be allowed to incorporate asymptotical ingredients, i.e. coming from high enough F_n (for an explicit or computable n .)

In this paper, we introduce as parameters some accessible norm subgroups of the pro- p -completion \bar{U}_F of the group of p -units of F , more precisely, the subgroup \tilde{U}_F of (global) universal norms in the cyclotomic \mathbb{Z}_p -extension F_∞/F , as well as the subgroup \hat{U}_F of those which are locally universal norms in the local cyclotomic \mathbb{Z}_p -extensions $F_{v,\infty}/F_v$ at all p -adic primes v (hence at all finite primes): $\tilde{U}_F \subset \hat{U}_F \subset \bar{U}_F$. It is known ([FN91, MS03, Se11]) that every element of \tilde{U}_F starts a \mathbb{Z}_p -extension, i.e. $\tilde{U}_F F^{\bullet p}/F^{\bullet p} \subset \mathcal{A}_F$. Our goal is to compare \mathcal{A}_F with various radicals derived from \tilde{U}_F and \hat{U}_F .

As a first step, we bound the Kummer radical \mathcal{A}_F "from below" by the radicals $\tilde{U}_F F^{\bullet p}/F^{\bullet p}$ and $\mathcal{A}_F \cap (\hat{U}_F/p)$ and describe the deviations in terms of some asymptotic capitulation kernels (Corollary 2.5 and Proposition 2.7). Then, we give an "upper bound" for \mathcal{A}_F in terms of the fixed points $(\tilde{U}_{F_n}/p)^{G_n}$ for $n \gg 0$ (but accessible) with $G_n = \text{Gal}(F_n/F)$, and describe the deviation between them (Theorem 3.1). Finally, we introduce a radical \mathcal{B}_F defined by conditions of \mathbb{Z}_p -embeddability locally everywhere, which contains the previous radicals \mathcal{A}_F , \mathcal{T}_F and \hat{U}_F/p , and we determine the three respective quotients in Iwasawa theoretical terms. Since the modules \tilde{U}_{F_n} (resp. the capitulation kernels) are immediately (resp. asymptotically and effectively) accessible, our goal has been reached.

Although our approach has a theoretical orientation, it lends itself to effective or algorithmic calculations, as will be shown in the examples of section 4, where we shall study the three radicals \mathcal{A}_F , \mathcal{T}_F and \hat{U}_F/p for $p = 3$ and biquadratic fields of the form $F = \mathbb{Q}(\mu_3, \sqrt{d})$.

We gather here the notations used in the text. Note that not all of them agree with the usual notations in Iwasawa theory as appearing for instance in Iwasawa's paper [Iw73].

F	our base number field;
F^\bullet	multiplicative group of non-zero elements of F ;
r_1, r_2	number of real (resp. non-conjugate complex) embeddings of F ;
$F_\infty = \cup F_n$	cyclotomic \mathbb{Z}_p -extension of F , with finite layers F_n ;
s, s_n	number of p -adic primes in F (resp. in F_n);
$\mu_{p^n}, \mu_{p^\infty}$	group of p^n -th (resp. all p -primary) roots of unity;
$\mu(F) = \mu_{p^\infty} \cap F$	group of all p -primary roots of unity contained in F ;
U_F, U_n	group of (p) -units in F (resp. in F_n);
$\bar{U}_F = \varprojlim (U_F/p^m), \bar{U}_n$	pro- p -completion of U_F (resp. of U_n);
F_v	completion of F at a prime v in F ;
\hat{U}_F, \hat{U}_n	subgroup of \bar{U}_F (resp. \bar{U}_n) consisting of elements which are universal norms from the cyclotomic \mathbb{Z}_p -extensions $F_{\infty,v}$ for all finite primes v ;
\tilde{U}_F, \tilde{U}_n	subgroup of \bar{U}_F (resp. \bar{U}_n) consisting of universal norms from F_∞ ;
V_n	factor group \bar{U}_n/\tilde{U}_n
$\bar{U}_\infty = \varprojlim \bar{U}_n$	inverse limit with respect to norm maps;
$\Lambda = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$	Iwasawa algebra, the isomorphism being obtained by mapping γ to $1 + T$;
S	set of all places of F over p ;
A_F, A_n	p -primary part of the (p) -class group of F (resp. of F_n);
$A_\infty = \varinjlim A_n$	p -primary part of the (p) -class group of F_∞ ;
L_n, L_∞	maximal unramified abelian pro- p -extension of F_n (resp. of F_∞) in which every prime over p splits completely;
\hat{F}	maximal abelian pro- p -extension of F unramified outside the p -adic primes;
F^{BP}	field of Bertrandias-Payan over F , <i>i.e.</i> , the compositum of all p -extensions of F which are infinitely embeddable in cyclic p -extensions;
\bar{F}_v^\bullet	pro- p -completion of F_v^\bullet ;
\tilde{F}_v^\bullet	group of universal norms in the cyclotomic \mathbb{Z}_p -extension $F_{\infty,v}/F_v$;
$\tilde{\mathcal{F}} = \prod \tilde{F}_v^\bullet$	
Γ, Γ_n	Galois group $\text{Gal}(F_\infty/F)$ (resp. $\text{Gal}(F_\infty/F_n)$);
γ	a fixed topological generator of Γ ;
$G_n \simeq \mathbb{Z}/p^n\mathbb{Z}$	Galois group $\text{Gal}(F_n/F)$;
BP_F	Galois group $\text{Gal}(F^{BP}/F)$;
$G_S(F)$	Galois group over F of the maximal S -ramified extension of F ;
\mathfrak{X}_F	Galois group over F of the maximal S -ramified abelian pro- p -extension of F ; $= G_S(F)^{\text{ab}} \otimes \mathbb{Z}_p$;
\mathfrak{X}_v	Galois group over F_v of the maximal abelian pro- p -extension of F_v ;
$\mathfrak{X}_\infty = \varprojlim \mathfrak{X}_{F_n}$	Galois group over F_∞ of the maximal S -ramified abelian pro- p -extension of F_∞ ;
\mathfrak{X}'_F	maximal factor group $(\mathfrak{X}_\infty)_\Gamma$ of \mathfrak{X}_∞ on which Γ acts trivially;
$X_\infty = \varprojlim A_n$	Galois group $\text{Gal}(L_\infty/F_\infty)$;
Δ_∞	Gross asymptotical defect, namely, $\varinjlim (X_\infty^{\Gamma_n} \otimes \mathbb{Q}_p/\mathbb{Z}_p)$;
X°	maximal finite submodule of X_∞ .

When μ_p is contained in F :

\mathcal{A}_F	the Kummer radical of the first layers of \mathbb{Z}_p -extensions of F ;
\mathcal{T}_F	subgroup of $F^\bullet/F^{\bullet p}$ consisting of classes $a \bmod F^{\bullet p}$ such that the symbol $\{\zeta_p, a\} = 0$ in K_2F ;
\mathcal{B}_F	subgroup of $F^\bullet/F^{\bullet p}$ consisting of classes $a \bmod F^{\bullet p}$ such that $F(\sqrt[p]{a})/F$ can be embedded in cyclic p -extensions of arbitrarily large degree;

The notation $(-)'$ for a group indicates that we have factored out the p -primary roots of unity.

Thus:

$$\begin{aligned}
\bar{U}'_F &= \bar{U}_F / \mu(F) && \text{group of } (p)\text{-units in } F \text{ factored by its torsion part;} \\
\bar{U}'_n &= \bar{U}_n / \mu_{p^\infty}(F_n) && \text{group of } (p)\text{-units in } F_n \text{ factored by its torsion part;} \\
U'_\infty &= \varinjlim U'_n && \text{group of } (p)\text{-units in } F_\infty; \\
\bar{U}'_\infty &= \varprojlim \bar{U}'_n && \text{inverse limit with respect to the norm maps;} \\
\mathcal{A}'_F &= \mathcal{A}_F / (\mu(F)/p) && \text{Kummer radical obtained by factoring out the cyclotomic} \\
&&& \mathbb{Z}_p\text{-extension of } F; \\
\mathcal{B}'_F &= \mathcal{B}_F / (\mu(F)/p) \\
\mathcal{T}'_F &= \mathcal{T}_F / (\mu(F)/p)
\end{aligned}$$

We will also need the following notations inside the "universal kummer radical" $\mathcal{K} = F_\infty^\bullet \otimes \mathbb{Q}_p / \mathbb{Z}_p$:

$$\begin{aligned}
\mathcal{L} &\cong \text{Hom}(\text{fr}_\Lambda(\mathcal{X}_\infty), \mu_{p^\infty}); \\
\mathfrak{N} &= \varinjlim (\bar{U}_n \otimes \mathbb{Q}_p / \mathbb{Z}_p); \\
\hat{\mathfrak{N}} &= \varinjlim (\hat{U}_n \otimes \mathbb{Q}_p / \mathbb{Z}_p); \\
\tilde{\mathfrak{N}} &= \varinjlim (\tilde{U}_n \otimes \mathbb{Q}_p / \mathbb{Z}_p);
\end{aligned}$$

If n is a non-negative integer and A is an abelian group, we denote by $A[n]$ the kernel of multiplication by n , and by A/n the cokernel. For a prime number p , we denote by $A\{p\}$ the p -primary part of A . Also $\text{Div}(A)$ will denote the maximal divisible subgroup of A and $A / \text{Div}(A)$ is simply written A / Div . If M is a module over a ring R , $\text{tor}_R(M)$ is the R -torsion sub-module of M , and $\text{fr}_R(M) := M / \text{tor}_R(M)$. Finally, $(-)^* = \text{Hom}(-, \mathbb{Q}_p / \mathbb{Z}_p)$ is the Pontrjagin dual.

1 Norm subgroups and a radical at infinite level

In this section, we recall (and prove if necessary) a number of results, which are fragmented and more or less well-known, concerning some norm subgroups of (p) -units and the Kummer radical of an Iwasawa module related to our problem. They will not all be needed in the sequel, but we will give as complete an account as possible, relying essentially on theorems of Kuz'min [Ku72]. If pressed for time, the reader can skip this section, coming back to it if necessary.

1.1 Global and local universal norms

Let U_F be the group of (p) -units in our number field F . Its pro- p -completion, denoted by \bar{U}_F , is $U_F \otimes \mathbb{Z}_p$ since U_F is finitely generated. Along the cyclotomic tower, \bar{U}_{F_n} is simply written \bar{U}_n and we put $\bar{U}_\infty = \varprojlim \bar{U}_n$ for norm maps. The \mathbb{Z}_p -torsion of \bar{U}_n is the group $\mu_{p^\infty}(F_n)$ of p -primary roots of unity contained in F_n . By adding an apostrophe we indicate the \mathbb{Z}_p -free part of our modules:

$$\bar{U}'_n := \bar{U}_n / \mu_{p^\infty}(F_n).$$

Finally, we put: $\bar{U}'_\infty := \varprojlim \bar{U}'_n$.

The following result, due to Kuz'min [Ku72], gives the Λ -structure of \bar{U}_∞ . It is classically known.

Proposition 1.1. *\bar{U}'_∞ is Λ -free of rank $r_1 + r_2$. When $\mu_p \subset F$ then we have a Λ -module isomorphism $\bar{U}_\infty \simeq \mathbb{Z}_p(1) \oplus \bar{U}'_\infty$.*

Definition 1.2. *The universal norm subgroup of \bar{U}_F (resp. of \bar{U}'_F) is the intersection $\cap_{n \geq 0} N_n(\bar{U}_n)$, denoted by \tilde{U}_F (resp. of $\cap_{n \geq 0} N_n(\bar{U}'_n)$, denoted by \tilde{U}'_F). Here N_n is the norm map in F_n/F .*

The usual compactness argument shows that \tilde{U}_F is the image of the natural map

$$(\bar{U}_\infty)_\Gamma \rightarrow \bar{U}_F.$$

Furthermore, this co-descent morphism is injective [Ku72, Theorem 7.3] so that $\tilde{U}_F \simeq (\bar{U}_\infty)_\Gamma$ is of \mathbb{Z}_p -rank equal to $r_1 + r_2$. Similarly, \tilde{U}'_F is isomorphic to $(\bar{U}'_\infty)_\Gamma$ and is of \mathbb{Z}_p -rank $r_1 + r_2$.

We are now going to define a second norm subgroup of \bar{U}_F . Let $\bar{F}_v^\bullet := \varprojlim \bar{F}_v^\bullet / F_v^{\bullet p^n}$ denote the pro- p -completion of F_v^\bullet , and \tilde{F}_v^\bullet denote the group of universal norms in the cyclotomic \mathbb{Z}_p -extension

of F_v (the local analogue of definition 1.2). Let $X_\infty := \text{Gal}(L_\infty/F_\infty)$, where L_∞ is the maximal unramified abelian pro- p -extension of F_∞ in which every prime over p (hence every prime) splits. We have the following Sinnott exact sequence (see [FGS81, Ja87, Ko91, etc]):

$$\bar{U}_F \xrightarrow{g_F} \tilde{\oplus}_{v|p} \bar{F}_v^\bullet / \tilde{F}_v^\bullet \xrightarrow{\text{Artin}} (X_\infty)_\Gamma \rightarrow A_F \rightarrow 0 \quad (\text{Sinnott})$$

where g_F is naturally deduced from the diagonal map taking its values in $\tilde{\oplus}_{v|p} \bar{F}_v^\bullet / \tilde{F}_v^\bullet$, the elements having a trivial sum of components.

Definition 1.3. *The everywhere local universal norm subgroup \hat{U}_F of \bar{U}_F consists of those elements which are locally universal norms in the cyclotomic \mathbb{Z}_p -extensions $F_{\infty,v}/F_v$ for all finite primes v in F . Since for a non- p -adic prime v , the cyclotomic \mathbb{Z}_p -extension F_v is unramified, the group of universal norms in $F_{\infty,v}/F_v$ is precisely the group of units of F_v . Therefore*

$$\hat{U}_F = \text{Ker } g_F.$$

We obviously have the following inclusions $\tilde{U}_F \subset \hat{U}_F \subset \bar{U}_F$, as well as $\tilde{U}'_F \subset \hat{U}'_F \subset \bar{U}'_F$, where an apostrophe indicates factoring out the \mathbb{Z}_p -torsion subgroup $\mu(F)$.

The next lemma, which is proved by class field theory, gives a characterization of the Gross kernel \hat{U}_F (see [BP72, Proposition 2.1] and [Ko91, section 1]).

Lemma 1.4. *For an element $x \in \bar{U}_F$, we have:*

$$x \in \hat{U}_F \iff N_{F_v/\mathbb{Q}_p}(x) \in p^\mathbb{Z} \quad \forall v|p.$$

This lemma implies that \hat{U}_F is "accessible" in the sense of the introduction. According to the above Sinnott exact sequence, the \mathbb{Z}_p -rank of \hat{U}_F is equal to $r_1 + r_2 + \delta$, where $\delta := rk_{\mathbb{Z}_p}(X_\infty)_\Gamma$. The number field F satisfies Gross' (generalized) Conjecture at p if $\delta = 0$, namely if $(X_\infty)_\Gamma$ (or equivalently X_∞^Γ , because X_∞ is Λ -torsion) is finite. Gross' Conjecture is known to hold for abelian extensions of \mathbb{Q} [Gre73, Ja87]. In the inclusion tower $\tilde{U}_F \subset \hat{U}_F \subset \bar{U}_F$, we have

$$\bar{U}_F / \hat{U}_F \simeq \text{im } g_F \simeq \mathbb{Z}_p^{s-1-\delta}$$

where s is the number of p -adic primes of F . Concerning the deviation between \hat{U}_F and \tilde{U}_F we have the following local-global result:

Proposition 1.5. *([Ku72, Proposition 7.5]) There exists a canonical exact sequence*

$$0 \rightarrow \tilde{U}_F \rightarrow \hat{U}_F \rightarrow X_\infty^\Gamma \rightarrow 0.$$

Kuz'min's proof is class field theoretic. For a proof with a more Iwasawa-theoretic flavour, see [KNF96, Theorem 3.3]. See also [Ka06, Section 3.2] for a cohomological proof. Our interest in \tilde{U}_F , as we recalled in the introduction, lies in the fact that, in the Kummerian situation, every element of \tilde{U}_F starts a \mathbb{Z}_p -extension. We give a quick proof for the convenience of the reader:

Lemma 1.6. *When $\mu_p \subset F$, the Kummer radical \mathcal{A}_F contains $\tilde{U}_F F^{\bullet p} / F^{\bullet p}$.*

Proof. Recall Kuz'min's result that the natural codescent map gives an isomorphism $(\bar{U}_\infty)_\Gamma \simeq \tilde{U}_F \subset \bar{U}_F \simeq H^1(G_S(F), \mathbb{Z}_p(1))$.

On the other hand, codescent on the (-1) -twist gives a homomorphism $\bar{U}_\infty(-1)_\Gamma \rightarrow \text{Hom}(G_S(F), \mathbb{Z}_p)$ (see [FN91, Theorem 3.7], where $NB(\mathcal{R}_F, \mathbb{Z}_p) \simeq \bar{U}_\infty(-1)_\Gamma$ and $G(\mathcal{R}_F, \mathbb{Z}_p) \simeq \text{Hom}(G_S(F), \mathbb{Z}_p)$). Hence a composite homomorphism

$$\bar{U}_\infty(-1)_\Gamma / p \rightarrow \text{Hom}(G_S(F), \mathbb{Z}_p) / p \xrightarrow{\text{nat}} \text{Hom}(G_S(F), \mathbb{Z}/p).$$

Since $\bar{U}_\infty(-1)_\Gamma / p = (\bar{U}_\infty)_\Gamma / p(-1) \simeq \tilde{U}_F / p(-1)$, it follows from Proposition 2.3, op. cit. that the induced map

$$(\tilde{U}_F / p)(-1) \longrightarrow \text{Hom}(G_S(F), \mathbb{Z}/p) = \text{Hom}(G_S(F), \mu_p)(-1)$$

is just the (-1) -twist of $\tilde{U}_F / p \longrightarrow \bar{U}_F / p$. As it factors through $\text{Hom}(G_S(F), \mathbb{Z}_p) / p(-1)$, the proof is complete. \square

1.2 A Kummer radical at infinite level

Let \mathcal{X}_∞ be the Galois group over F_∞ of the maximal abelian p -extension of F_∞ which is unramified outside p -adic primes. It is known that the Λ -rank of \mathcal{X}_∞ is equal to r_2 (this is the weak Leopoldt conjecture, which holds in the case of the cyclotomic \mathbb{Z}_p -extension [Wa97, Section 13.5]). Put $\text{fr}_\Lambda(\mathcal{X}_\infty)$ for its torsion-free part. When $\mu_p \subset F$, the Kummer radical of $\text{fr}_\Lambda(\mathcal{X}_\infty)$ is clearly related to the problem we are interested in. The determination of this Kummer radical has been performed independently by Kuz'min ([Ku72]) and Kolster ([Ko91]) using idelic methods. Here, we are going to prove a slightly more precise version of their result using a direct approach.

First fix the following notations: $\mathfrak{L} \cong \text{Hom}(\text{fr}_\Lambda(\mathcal{X}_\infty), \mu_{p^\infty}) \subseteq F_\infty^\bullet \otimes \mathbb{Q}_p/\mathbb{Z}_p$,
 $\mathfrak{N} = \varinjlim (\tilde{U}_n \otimes \mathbb{Q}_p/\mathbb{Z}_p)$, $\hat{\mathfrak{N}} = \varinjlim (\hat{U}_n \otimes \mathbb{Q}_p/\mathbb{Z}_p)$, $\tilde{\mathfrak{N}} = \varinjlim (\tilde{U}_n \otimes \mathbb{Q}_p/\mathbb{Z}_p)$.

For all $n \geq 0$, define V_n to fit into the following tautological short exact sequence

$$0 \rightarrow \tilde{U}_n \rightarrow \bar{U}_n \rightarrow V_n \rightarrow 0.$$

We then have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{U}_n & \longrightarrow & \hat{U}_n & \longrightarrow & X_\infty^{\Gamma_n} \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & \tilde{U}_n & \longrightarrow & \bar{U}_n & \longrightarrow & V_n \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & \tilde{\oplus}_{v|p} \bar{F}_v^\bullet / \tilde{F}_v^\bullet \simeq \mathbb{Z}_p^{s_n-1} & & \\ & & & & \downarrow & & \\ & & & & (X_\infty)_{\Gamma_n} & & \\ & & & & \downarrow & & \\ & & & & A_n & & \end{array}$$

where s_n is the number of p -adic primes in the layer F_n . This immediately provides a short exact sequence

$$0 \longrightarrow X_\infty^{\Gamma_n} \longrightarrow V_n \longrightarrow \mathbb{Z}_p^{s_n-1-\delta_n} \rightarrow 0$$

where $\delta_n := rk_{\mathbb{Z}_p}(X_\infty)_{\Gamma_n}$. In particular $\text{tor}_{\mathbb{Z}_p}(V_n) \simeq \text{tor}_{\mathbb{Z}_p}(X_\infty^{\Gamma_n})$.

Note that $\varinjlim \text{tor}_{\mathbb{Z}_p}(X_\infty^{\Gamma_n}) = X^\circ$ is the maximal finite submodule of X_∞ and the groups $X_\infty^{\Gamma_n} \otimes \mathbb{Q}_p/\mathbb{Z}_p$ stabilize. Put $\Delta_\infty := \varinjlim (X_\infty^{\Gamma_n} \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ for the "Gross asymptotical defect".

Proposition 1.7. ([Ko91, Ku72]) *We have an exact sequence of Γ -modules*

$$\begin{array}{ccccccc} 0 & \longrightarrow & X^\circ & \longrightarrow & \tilde{\mathfrak{N}} & \longrightarrow & \hat{\mathfrak{N}} \longrightarrow \Delta_\infty \longrightarrow 0 \\ & & & & \searrow & & \nearrow \\ & & & & \mathfrak{L} & & \end{array}$$

In particular, $\mathfrak{L} = \hat{\mathfrak{N}}$ precisely when all the layers F_n verify Gross' conjecture.

Proof. Tensoring with $\mathbb{Q}_p/\mathbb{Z}_p$ the exact sequence $0 \rightarrow \tilde{U}_n \rightarrow \hat{U}_n \rightarrow X_\infty^{\Gamma_n} \rightarrow 0$ of Proposition 1.5 we get:

$$0 \rightarrow \text{tor}_{\mathbb{Z}_p}(X_\infty^{\Gamma_n}) \rightarrow \tilde{U}_n \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \hat{U}_n \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow X_\infty^{\Gamma_n} \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0.$$

Hence, passing to the direct limit over n yields

$$0 \rightarrow X^\circ \rightarrow \tilde{\mathfrak{N}} \rightarrow \hat{\mathfrak{N}} \rightarrow \Delta_\infty \rightarrow 0$$

where X° is the maximal finite submodule of X_∞ . Now it remains to show that the cokernel of the first map $X^\circ \rightarrow \mathfrak{N}$ is precisely \mathfrak{L} . Proceeding in the same way as before from the exact sequence

$$0 \rightarrow \tilde{U}_n \rightarrow \bar{U}_n \rightarrow V_n \rightarrow 0$$

we get:

$$0 \rightarrow X^\circ \rightarrow \tilde{\mathfrak{N}} \rightarrow \mathfrak{N} \rightarrow \mathfrak{V} := \varinjlim (V_n \otimes \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow 0$$

which we split into two short exact sequences:

$$0 \rightarrow X^\circ \rightarrow \tilde{\mathfrak{N}} \rightarrow \mathfrak{L}' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathfrak{L}' \rightarrow \mathfrak{N} \rightarrow \mathfrak{V} \rightarrow 0.$$

Write the Kummer dual of the first sequence:

$$0 \rightarrow \text{Hom}(\mathfrak{L}', \mu_{p^\infty}) \rightarrow \text{Hom}(\tilde{\mathfrak{N}}, \mu_{p^\infty}) \rightarrow \text{Hom}(X^\circ, \mu_{p^\infty}) \rightarrow 0.$$

By definition, $\tilde{U}_n = (\bar{U}_\infty)_{\Gamma_n}$. In view of the properties of \bar{U}_∞ (Proposition 1.1), the module $\tilde{\mathfrak{N}}$ is a co-free module over Λ of co-rank r_2 : $\text{Hom}(\tilde{\mathfrak{N}}, \mu_{p^\infty}) \simeq \Lambda^{r_2}$. Therefore $\text{Hom}(\mathfrak{L}', \mu_{p^\infty})$ is Λ -torsion-free of the same rank r_2 . Now, the Kummer dual of the second sequence:

$$0 \rightarrow \text{Hom}(\mathfrak{V}, \mu_{p^\infty}) \rightarrow \text{Hom}(\mathfrak{N}, \mu_{p^\infty}) \rightarrow \text{Hom}(\mathfrak{L}', \mu_{p^\infty}) \rightarrow 0$$

shows that $\text{Hom}(\mathfrak{L}', \mu_{p^\infty})$ is a quotient module of $\text{Hom}(\mathfrak{N}, \mu_{p^\infty})$, and it is known that $\text{fr}_\Lambda \mathfrak{X}_\infty$ is the maximal Λ -torsion free quotient of the Galois group $\text{Hom}(\mathfrak{N}, \mu_{p^\infty})$ ([Iw73, Theorem 15]). Accordingly, we have a surjective map $\text{fr}_\Lambda \mathfrak{X}_\infty \rightarrow \text{Hom}(\mathfrak{L}', \mu_{p^\infty})$, which must be an isomorphism since they both have Λ -rank r_2 . Hence $\mathfrak{L}' = \mathfrak{L}$, as was to be shown. \square

Remark: The structure theorem for finitely generated Λ -modules shows the existence of a finite module H such that

$$0 \rightarrow \text{fr}_\Lambda \mathfrak{X}_\infty \rightarrow \Lambda^{r_2} \rightarrow H \rightarrow 0.$$

The above proof provides such an exact sequence in a canonical way as well as an isomorphism $H \simeq \text{Hom}(X^\circ, \mu_{p^\infty})$. This Kummer duality between H and X° was already implicit in [Iw73]. It is also known (op. cit.) that X° is isomorphic to the (asymptotical) capitulation kernel $\text{Ker}(A_m \rightarrow \varinjlim A_n)$ for m large [Gre78, page 1240].

2 Lower bounds for the Kummer radical \mathcal{A}_F

In this section we suppose that F contains μ_p . In order to get information about the radical \mathcal{A}_F , we are first going to do "descent" from the module \mathfrak{L} in the same way as in [Gre78] (but we will need somewhat more precise results). Recall the notations: $\tilde{U}'_F = \tilde{U}_F/\mu(F)$, $\hat{U}'_F = \hat{U}_F/\mu(F)$ and let $\mathcal{A}'_F := \mathcal{A}_F/(\mu(F)/p)$, where $\mu(F)/p \simeq \mu(F)F^{\bullet p}/F^{\bullet p}$ is the Kummer radical of the first layer of the cyclotomic \mathbb{Z}_p -extension of F .

2.1 A lower bound in terms of universal norms

The starting point will be the exact sequence

$$0 \rightarrow X^\circ \rightarrow \tilde{\mathfrak{N}} \rightarrow \mathfrak{L} = \mathfrak{L}' \rightarrow 0$$

of Proposition 1.7, where we recall that $\tilde{\mathfrak{N}} = \varinjlim (\tilde{U}_n \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ and $\mathfrak{L} \cong \text{Hom}(\text{fr}_\Lambda(\mathcal{X}_\infty), \mu_{p^\infty}) \subseteq F_\infty^\bullet \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

For all rational integers i , we consider the i -fold Tate-twists:

$$0 \rightarrow X^\circ(i) \rightarrow \tilde{\mathfrak{N}}(i) \rightarrow \mathfrak{L}(i) \rightarrow 0. \quad (1)$$

Theorem 2.1. *For $i \in \mathbb{Z}$, we have an exact sequence:*

$$0 \rightarrow X^{\circ\Gamma}[p](i) \rightarrow \tilde{U}'_F/p(i) \rightarrow \text{Div}(\mathfrak{L}(i)^\Gamma)[p] \rightarrow X^\circ(i)^\Gamma/p \rightarrow 0$$

where $\text{Div}(-)$ denotes the maximal divisible subgroup of $(-)$.

Notice that, since $\mu_p \subset F$, the twist i outside is purely cosmetic for the Galois action above F , but of course not for the action below F .

Proof. The exact sequence (1) provides us with:

$$0 \rightarrow X^\circ(i)^\Gamma \rightarrow \tilde{\mathfrak{N}}(i)^\Gamma \rightarrow \mathfrak{L}(i)^\Gamma \rightarrow X^\circ(i)_\Gamma \rightarrow \tilde{\mathfrak{N}}(i)_\Gamma \rightarrow \dots$$

Let N_i be the cokernel of the first map on the left: $0 \rightarrow X^\circ(i)^\Gamma \rightarrow \tilde{\mathfrak{N}}(i)^\Gamma \rightarrow N_i \rightarrow 0$. As noticed before (Proposition 1.1), the Λ -module $\tilde{\mathfrak{N}}$ is cofree. Hence $\tilde{\mathfrak{N}}(i)_\Gamma$ is trivial whereas $\tilde{\mathfrak{N}}(i)^\Gamma$ is divisible and therefore $N_i = \text{Div}(\mathfrak{L}(i)^\Gamma)$. Consequently, we have:

$$0 \rightarrow X^\circ(i)^\Gamma \rightarrow \tilde{\mathfrak{N}}(i)^\Gamma \rightarrow \text{Div}(\mathfrak{L}(i)^\Gamma) \rightarrow 0.$$

Applying the snake lemma to multiplication-by- p gives

$$0 \rightarrow X^\circ(i)^\Gamma[p] = X^{\circ\Gamma}[p](i) \rightarrow \tilde{\mathfrak{N}}(i)^\Gamma[p] = \tilde{\mathfrak{N}}^\Gamma[p](i) \rightarrow \text{Div}(\mathfrak{L}(i)^\Gamma)[p] \rightarrow X^\circ(i)^\Gamma/p \rightarrow 0.$$

To finish the proof of the Theorem, it remains to recognize $\tilde{\mathfrak{N}}^\Gamma$:

Lemma 2.2. $\tilde{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \simeq \tilde{\mathfrak{N}}^\Gamma$.

Proof. At finite levels, we have $\hat{U}'_n \hookrightarrow \hat{U}'_{n+1}$ (this is immediate from Lemma 1.4). Hence we also have $\tilde{U}'_n \hookrightarrow \tilde{U}'_{n+1}$. By definition 1.2, the norm maps $\tilde{U}'_n \rightarrow \tilde{U}'_F$ are surjective. On the other hand, by Proposition 1.1, all the \tilde{U}'_n are free \mathbb{Z}_p -modules. Hence \tilde{U}'_F is a direct summand of \tilde{U}'_n and the cokernel of the natural injection $\tilde{U}'_F \rightarrow \tilde{U}'_n$ is torsion-free. Accordingly, the maps $\tilde{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \tilde{U}'_n \otimes \mathbb{Q}_p/\mathbb{Z}_p$ are injective and so is the map: $\tilde{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \tilde{\mathfrak{N}}^\Gamma$.

Moreover, as previously explained, $\tilde{\mathfrak{N}}^\Gamma$ is divisible and to show the equality all we need to do is compare the \mathbb{Z}_p -coranks of these two modules: the exact sequence

$$0 \rightarrow X^{\circ\Gamma} \rightarrow \tilde{\mathfrak{N}}^\Gamma \rightarrow \text{Div}(\mathfrak{L}^\Gamma) \rightarrow 0$$

shows that $\text{corank}_{\mathbb{Z}_p}(\tilde{\mathfrak{N}}^\Gamma) = r_2$. On the other hand:

$$\begin{aligned} \text{rank}_{\mathbb{Z}_p}(\tilde{U}'_F) &= \text{rank}_{\mathbb{Z}_p}(\hat{U}'_F) - \text{rank}_{\mathbb{Z}_p}(X_\infty)^\Gamma && (Kuz'min) \\ &= \text{rank}_{\mathbb{Z}_p}(\tilde{U}'_F) - (s-1) + \text{rank}_{\mathbb{Z}_p}(X_\infty)_\Gamma - \text{rank}_{\mathbb{Z}_p}(X_\infty)^\Gamma && (Sinnott) \\ &= r_2 + \text{rank}_{\mathbb{Z}_p}(X_\infty)_\Gamma - \text{rank}_{\mathbb{Z}_p}(X_\infty)^\Gamma, \end{aligned}$$

where s is the number of p -adic primes in F . Besides, since X_∞ is a Λ -torsion module, denoting by Y_∞ the "non- μ -part" of X_∞ , we have a "pseudo-exact" sequence

$$0 \rightarrow (X_\infty)^\Gamma \rightarrow Y_\infty \xrightarrow{\gamma-1} Y_\infty \rightarrow (X_\infty)_\Gamma \rightarrow 0$$

of finitely generated \mathbb{Z}_p -modules which shows that $(X_\infty)^\Gamma$ and $(X_\infty)_\Gamma$ have the same \mathbb{Z}_p -rank. Hence finally the \mathbb{Z}_p -corank of $\tilde{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is also r_2 . \square

We now propose to give interpretations of $\text{Div}(\mathfrak{L}(i)^\Gamma)[p]$ in terms of radicals in the three special cases $i = -1, 0, 1$. We have already introduced $\mathcal{A}'_F := \mathcal{A}_F/(\mu(F)/p)$ and $(\hat{U}'_F/p) = (\tilde{U}'_F/p)/(\mu(F)/p)$, where $\mu(F)/p \cong \mu(F)F^{\bullet p}/F^{\bullet p}$ is the image of μ_{p^∞} in $F^{\bullet}/F^{\bullet p}$. We also want to introduce a modified tate kernel \mathcal{T}'_F . Using elementary properties of symbols, one easily shows that $\mu(F)F^{\bullet p}/F^{\bullet p}$ is also contained in \mathcal{T}_F , thus we define $\mathcal{T}'_F := \mathcal{T}_F/(\mu(F)/p)$.

Proposition 2.3. *We have the following equalities:*

- (a) *if $i = -1$ and we assume Leopoldt's conjecture for F , then $\text{Div}(\mathfrak{L}(-1)^\Gamma)[p](1) = \mathcal{A}'_F$.*
- (b) *if $i = 0$ and we assume Gross' conjecture for F , then $\text{Div}(\mathfrak{L}^\Gamma)[p] = \hat{U}'_F/p$.*
- (c) *if $i = 1$, then $\text{Div}(\mathfrak{L}(1)^\Gamma)[p](-1) = \mathcal{T}'_F$.*

Recall that if Gross' conjecture is valid for all the F_n 's, then $\mathfrak{L} = \hat{\mathfrak{N}}$.

Proof. Properties (a) and (c) have been explained by Greenberg, starting from his conjecture ([Gre78, page 1238]; see also [Sc79, page 192]) that for all $i \neq 0$, the co-rank of $\text{Div}(\mathcal{K}(i)^\Gamma)$ should be r_2 , hence $\text{Div}(\mathcal{K}(i)^\Gamma) = \text{Div}(\mathfrak{L}(i)^\Gamma)$ where $\mathcal{K} := F_\infty^\bullet \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong H^1(F_\infty, \mu_{p^\infty})$.

For $i = -1$ this is Leopoldt's conjecture and for $i = 1$ a consequence of results of Tate on K_2 (see [Co72, Theorem 2]). For the convenience of the reader, we reprove (a) and (c) along our own lines. For (a), let \hat{F} be the maximal abelian p -extension of F which is unramified outside p -adic primes and $\mathfrak{X}_F := \text{Gal}(\hat{F}/F)$. Put $\mathfrak{X}'_F := (\mathfrak{X}_\infty)_\Gamma = \text{Gal}(\hat{F}/F_\infty)$. Then, the following exact sequence of co-descent

$$0 \rightarrow \mathfrak{X}'_F \rightarrow \mathfrak{X}_F \rightarrow \text{Gal}(F_\infty/F) \simeq \mathbb{Z}_p \rightarrow 0$$

shows, under Leopoldt's conjecture, that $\text{tor}_{\mathbb{Z}_p}(\mathfrak{X}'_F) = \text{tor}_{\mathbb{Z}_p}(\mathfrak{X}_F)$ and the exactness of

$$0 \rightarrow \text{fr}_{\mathbb{Z}_p} \mathfrak{X}'_F \rightarrow \text{fr}_{\mathbb{Z}_p} \mathfrak{X}_F \rightarrow \mathbb{Z}_p \rightarrow 0.$$

Consequently, our \mathcal{A}'_F is the Kummer radical of $(\text{fr}_{\mathbb{Z}_p} \mathfrak{X}'_F)/p$. Besides, the co-descent from

$$0 \rightarrow \text{tor}_\Lambda \mathfrak{X}_\infty \rightarrow \mathfrak{X}_\infty \rightarrow \text{fr}_\Lambda \mathfrak{X}_\infty \rightarrow 0$$

yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\text{tor}_\Lambda \mathfrak{X}_\infty)_\Gamma & \longrightarrow & (\mathfrak{X}_\infty)_\Gamma = \mathfrak{X}'_F & \longrightarrow & (\text{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma \longrightarrow 0 \\ & & \downarrow \epsilon_1 & & \parallel & & \downarrow \epsilon_2 \\ 0 & \longrightarrow & \text{tor}_{\mathbb{Z}_p}(\mathfrak{X}_F) & \longrightarrow & \mathfrak{X}'_F & \longrightarrow & \text{fr}_{\mathbb{Z}_p} \mathfrak{X}'_F \longrightarrow 0 \end{array}$$

Leopoldt's conjecture for F is equivalent to the finiteness of $(\text{tor}_\Lambda \mathfrak{X}_\infty)_\Gamma$. This ensures that the vertical map ϵ_1 takes its values in $\text{tor}_{\mathbb{Z}_p}(\mathfrak{X}_F)$. Hence the existence of the map ϵ_2 and the isomorphism $\text{Ker } \epsilon_2 \simeq \text{Coker } \epsilon_1$. In particular, $\text{Ker } \epsilon_2$ is finite and therefore $\text{fr}_{\mathbb{Z}_p}((\text{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma) \simeq \text{fr}_{\mathbb{Z}_p}(\mathfrak{X}'_F)$.

Write the exact sequence of the Remark following Proposition 1.7:

$$0 \rightarrow \text{fr}_\Lambda \mathfrak{X}_\infty \rightarrow \Lambda^{r_2} \rightarrow H \rightarrow 0$$

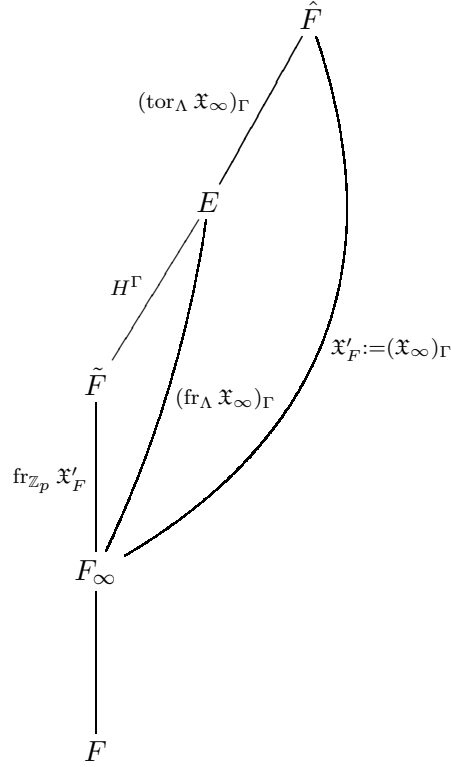
where the finite module H is isomorphic to $\text{Hom}(X^\circ, \mu_{p^\infty})$. By codescent, we obtain

$$0 \rightarrow H^\Gamma \rightarrow (\text{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma \rightarrow \mathbb{Z}_p^{r_2}$$

which shows that: $H^\Gamma \simeq \text{tor}_{\mathbb{Z}_p}((\text{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma)$, hence the following exact sequence

$$0 \rightarrow H^\Gamma \rightarrow (\text{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma \xrightarrow{\epsilon_2} \text{fr}_{\mathbb{Z}_p}(\mathfrak{X}'_F) \rightarrow 0.$$

The situation is illustrated in the following diagram of fields and Galois groups:



The above short exact sequence gives by Kummer duality

$$0 \rightarrow \text{Hom}(\text{fr}_{\mathbb{Z}_p} \mathfrak{X}'_F, \mu_{p^\infty}) \rightarrow \mathfrak{L}(-1)^\Gamma(1) \rightarrow (X^\circ(-1)_\Gamma)(1) \rightarrow 0$$

where, as before, $\mathfrak{L} = \text{Hom}(\text{fr}_\Lambda \mathfrak{X}_\infty, \mu_{p^\infty})$. This shows that the first term is $\text{Div}(\mathfrak{L}(-1)^\Gamma)(1)$ and the snake lemma gives the exact sequence

$$0 \rightarrow \mathcal{A}'_F \rightarrow \mathfrak{L}(-1)^\Gamma[p](1) = \mathfrak{L}^\Gamma[p] \rightarrow (X^\circ(-1)_\Gamma)[p](1) \rightarrow 0$$

(a precise version of the result in [Gre78, page 1242]). This shows in particular that under Leopoldt's conjecture $\mathcal{A}'_F = \text{Div}(\mathfrak{L}(-1)^\Gamma)[p](1)$.

To show (c), start from the commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \text{Div}(\mathcal{K}(1)^\Gamma)[p] & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & (\mu(F)/p)(1) & \longrightarrow & F^\bullet/F^{\bullet p}(1) & \longrightarrow & \mathcal{K}(1)^\Gamma[p] = \mathcal{K}^\Gamma[p](1) \longrightarrow 0 \\
 & & \downarrow & & \downarrow = & & \downarrow \beta \\
 0 & \longrightarrow & \mathcal{T}_F(1) & \longrightarrow & F^\bullet/F^{\bullet p}(1) & \xrightarrow{\alpha} & K_2 F[p] \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

The upper exact line comes from Kummer theory: just write the cohomology exact sequence

$$0 \rightarrow H^0(F, \mu_{p^\infty})/p \rightarrow H^1(F, \mu_p) \rightarrow H^1(F, \mu_{p^\infty})[p] \rightarrow 0$$

and use the nullity of $H^1(\Gamma, \mu_{p^\infty})$ (Tate's lemma) to show that

$$H^1(F, \mu_{p^\infty}) \cong H^1(F_\infty, \mu_{p^\infty})^\Gamma \cong \mathcal{K}^\Gamma.$$

The map α in the lower exact line is Tate's map $a \otimes \zeta_p \mapsto \{a, \zeta_p\}$ which defines the Tate kernel \mathcal{T}_F . The exact column is derived from Tate's exact sequence (see e.g. [Co72, Theorem 3])

$$0 \rightarrow \text{Div}(\mathcal{K}(1)^\Gamma) \rightarrow \mathcal{K}(1)^\Gamma \xrightarrow{\beta} K_2 F\{p\} \rightarrow 0$$

where β is the map $a \otimes \zeta \mapsto \{a, \zeta\}$, for $a \in F_\infty^\bullet$ and $\zeta \in \mu_{p^\infty}$. A simple diagram chase now gives (c).

Greenberg's conjecture does not concern the twist $i = 0$, since \mathcal{K}^Γ is of infinite co-type. Hence the assertion (b) requires special treatment:

Lemma 2.4. *Assuming Gross' conjecture for F , we have an exact sequence*

$$0 \rightarrow \hat{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathfrak{L}^\Gamma \rightarrow (X^\circ)_\Gamma \rightarrow 0.$$

In particular, $\hat{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p = \text{Div}(\mathfrak{L}^\Gamma)$.

Proof. We have two short exact sequences (Proposition 1.7 and its proof)

$$0 \rightarrow X^\circ \rightarrow \tilde{\mathfrak{N}} \rightarrow \mathfrak{L} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathfrak{L} \rightarrow \hat{\mathfrak{N}} \rightarrow \Delta_\infty \rightarrow 0.$$

From the first, we have already derived:

$$0 \rightarrow (X^\circ)^\Gamma \rightarrow \tilde{\mathfrak{N}}^\Gamma \rightarrow \mathfrak{L}^\Gamma \rightarrow (X^\circ)_\Gamma \rightarrow 0.$$

Since \mathfrak{L} is the Kummer dual of $\text{fr}_\Lambda(\mathfrak{X}_\infty)$, we have $\mathfrak{L}_\Gamma = 0$ so that the second sequence implies:

$$0 \rightarrow \mathfrak{L}^\Gamma \rightarrow \hat{\mathfrak{N}}^\Gamma \rightarrow \Delta_\infty^\Gamma \rightarrow 0.$$

We compare this exact sequence to what we have at the level of F which is provided by Proposition 1.5 after tensoring with $\mathbb{Q}_p/\mathbb{Z}_p$:

$$0 \rightarrow \text{tor}_{\mathbb{Z}_p}(X_\infty^\Gamma) \rightarrow \tilde{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \hat{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow X_\infty^\Gamma \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0.$$

Namely, we break the above exact sequence into two exact sequences to obtain a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (X^\circ)^\Gamma & \longrightarrow & \tilde{\mathfrak{N}}^\Gamma & \longrightarrow & \mathfrak{L}^\Gamma \longrightarrow (X^\circ)_\Gamma \longrightarrow 0 \\ & & \parallel & & \uparrow \wr & & \uparrow \\ 0 & \longrightarrow & \text{tor}_{\mathbb{Z}_p}(X_\infty^\Gamma) & \longrightarrow & \tilde{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & M \longrightarrow 0 \end{array}$$

leading to

$$0 \rightarrow M \rightarrow \mathfrak{L}^\Gamma \rightarrow (X^\circ)_\Gamma \rightarrow 0.$$

To finish the proof, observe that under the Gross conjecture for F , the tensor product $X_\infty^\Gamma \otimes \mathbb{Q}_p/\mathbb{Z}_p$ vanishes since X_∞^Γ is finite. Hence in fact $M = \hat{U}'_F \otimes \mathbb{Q}_p/\mathbb{Z}_p$. \square

Since we are specifically dealing with the case (a) of the above Proposition, let us restate it after twisting once "à la Tate".

Corollary 2.5. *If F verifies Leopoldt's conjecture at p , then we have a short exact sequence*

$$0 \rightarrow \tilde{U}_F F^{\bullet p}/F^{\bullet p} \rightarrow \mathcal{A}_F \rightarrow (X^\circ(-1)^\Gamma/p)(1) \rightarrow 0.$$

In particular, $\mathcal{A}_F = \tilde{U}_F F^{\bullet p}/F^{\bullet p}$ precisely when $X^\circ = 0$.

Proof. The case $i = -1$ of Theorem 2.1 yields:

$$0 \rightarrow \tilde{U}'_F F^{\bullet p} / F^{\bullet p} \rightarrow \mathcal{A}'_F \rightarrow (X^\circ(-1)^\Gamma / p)(1) = (X_\infty(-1)^\Gamma / p)(1) \rightarrow 0$$

(The last equality comes from Leopoldt's conjecture, which is known to be equivalent to the finiteness of $X_\infty(-1)^\Gamma$). Since $\tilde{U}'_F F^{\bullet p} / F^{\bullet p}$ and \mathcal{A}'_F are obtained by taking the quotients of $\tilde{U}_F F^{\bullet p} / F^{\bullet p}$ and \mathcal{A}_F by the same submodule $\mu(F)/p$, the exact sequence of the corollary follows. \square

Example 2.6. Consider a number field F which satisfies Leopoldt's conjecture together with the following two properties:

(i) $X^\circ = (0)$ and (ii) F contains only one p -adic prime (for a cyclotomic field $F := \mathbb{Q}(\mu_{p^r})$, the property (i) is equivalent to Vandiver's conjecture and (ii) is of course automatically satisfied). Then $\mathcal{A}_F = \hat{U}_F F^{\bullet p} / F^{\bullet p}$ according to Proposition 1.5 and Corollary 2.5, and $\bar{U}_F = \hat{U}_F$ according to (ii). It follows that $\mathcal{A}_F = \bar{U}_F / p$.

Remarks: (i) Lemma 2.4 was shown in [Hu05] using a different approach.

(ii) Within the context of assertion (b), the exact sequence of Theorem 2.1 coincides with the one obtained by applying the snake lemma to multiplication-by- p in Kuz'min's exact sequence (Proposition 1.5).

(iii) For $i \neq 0$, the exact sequence of Theorem 2.1 comes also by applying the snake lemma to multiplication-by- p in some descent exact sequences in Galois cohomology [KNF96, Theorem 3.2 bis].

(iv) In Corollary 2.5, $\tilde{U}'_F F^{\bullet p} / F^{\bullet p}$ is the Kummer radical of the first layers of \mathbb{Z}_p -extensions of a particular type. One can show from [FN91, Theorem 2.4] that they are the \mathbb{Z}_p -extensions $K_\infty = \cup_{n \geq 0} K_n$ of F , such that the rings of p -integers of all the K_n 's have normal bases which are coherent for the trace.

(v) For $i \neq 0$, the G_F -modules $\text{Div}(\mathfrak{L}(i)^\Gamma)[p]$ are the generalized Tate kernels studied in [?, AM04, Hu05, Va08] in connection with problems of capitulation. When $F \supset \mu_{p^{e+1}}$, where p^e is the exponent of X° , then all the modules $\text{Div}(\mathfrak{L}(i)^\Gamma)[p]$ are equal [Gre78, AM04] (see also Corollary 3.2 below). For a general comparison between these kernels when i varies see [Va08, Theorem 2.7].

Under Leopoldt's conjecture, the above Corollary 2.5 provides a good approximation of \mathcal{A}_F by $\tilde{U}_F F^{\bullet p} / F^{\bullet p}$, whose dimension over \mathbb{F}_p is $r_2 - h$, where $h := \dim X^{\circ \Gamma}[p] = \dim X^\circ(-1)^\Gamma[p] = \dim(X^\circ(-1)^\Gamma / p)$. The difference h is of an asymptotic nature and bounded in the cyclotomic tower. Interpreting X° as a capitulation kernel [Iw73, Gre78], this parameter is theoretically and effectively accessible.

Nevertheless, the result is not entirely satisfactory as the p -units of \tilde{U}_F / p are not immediately accessible. Kuz'min's exact sequence (Proposition 1.5) suggests replacing \tilde{U}_F by \hat{U}_F , which is easily accessible by Lemma 1.4.

2.2 A lower bound in terms of local universal norms

We want to "approximate" \mathcal{A}_F by the intersection $\mathcal{A}_F \cap \hat{U}_F / p$. In order to compute the deviation, let us come back to the exact sequence of Theorem 2.1, where the map $\sigma_i : \text{Div}(\mathfrak{L}(i)^\Gamma)[p] \rightarrow X^\circ(i)^\Gamma / p$ is given by the snake lemma and hence depends on the twist i . To compare the images of the σ_i 's, we must put them in a space which does not (at least for the action of Galois groups over F) depend on i . From the exact sequence

$$0 \rightarrow X^\circ(i)^\Gamma \rightarrow \mathfrak{N}(i)^\Gamma \rightarrow \mathfrak{L}(i)^\Gamma$$

(see Proposition 1.7), we derive

$$0 \rightarrow (X^\circ)^\Gamma[p](i) \rightarrow \mathfrak{N}^\Gamma[p](i) \rightarrow \mathfrak{L}^\Gamma[p](i).$$

Let W be the cokernel of the map on the right, so that W does not depend on i (for the Galois action over F) and we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (X^\circ)^\Gamma[p](i) & \longrightarrow & \tilde{\mathfrak{H}}^\Gamma[p](i) & \longrightarrow & \mathfrak{L}^\Gamma[p](i) & \longrightarrow & W & \longrightarrow & 0 \\
& & \parallel & & \parallel & & \uparrow & & \uparrow \tau_i & & \\
0 & \longrightarrow & (X^\circ)^\Gamma[p](i) & \longrightarrow & \tilde{\mathfrak{H}}^\Gamma[p](i) & \longrightarrow & \text{Div}(\mathfrak{L}(i)^\Gamma)[p] & \xrightarrow{\sigma_i} & X^\circ(i)^\Gamma/p & \longrightarrow & 0
\end{array}$$

where the right vertical map τ_i is defined tautologically and is injective. Finally, let $T_i := \tau_i(X^\circ(i)^\Gamma/p)(-i)$, then with these notations we have

Proposition 2.7. *Suppose that Leopoldt's and Gross' conjecture are valid for F . Then we have the following exact sequence:*

$$0 \rightarrow \mathcal{A}_F \cap \hat{U}_F/p \rightarrow \mathcal{A}_F \rightarrow T_{-1}/T_0 \cap T_{-1} \rightarrow 0.$$

Proof. Write $D_i := \text{Div} \mathfrak{L}(i)^\Gamma[p]$ for short. Then $D_i(-i)$, for $i = 0, -1$, can be identified with the Kummer radicals in $F^\bullet/F^{\bullet p}$ of the Kummer extensions $F(\sqrt[p]{\hat{U}_F})/F$ and $F(\sqrt[p]{\mathcal{A}_F})/F$ respectively (obvious notations). According to Theorem 2.1, $\tau_i(X^\circ(i)^\Gamma/p)$ for $i = 0, -1$ can be identified with the Kummer radicals in $F(\sqrt[p]{\hat{U}_F})$ of $F(\sqrt[p]{\hat{U}_F})$ and $F(\sqrt[p]{\mathcal{A}_F})$ respectively (see the diagram). Hence the statement of the proposition by elementary Kummer theory.

$$\begin{array}{ccc}
F(\sqrt[p]{\mathcal{A}_F}) & \longrightarrow & F(\sqrt[p]{\mathcal{A}_F \hat{U}_F}) \\
\downarrow & & \downarrow \\
F(\sqrt[p]{\hat{U}_F}) \cap F(\sqrt[p]{\mathcal{A}_F}) & \longrightarrow & F(\sqrt[p]{\hat{U}_F}) \\
\downarrow & & \\
F(\sqrt[p]{\hat{U}_F}) & & \\
\downarrow & & \\
F & &
\end{array}$$

□

Remark. An analogous result holds when replacing the pair $(\mathcal{A}_F, \hat{U}_F/p)$ by any pair taken from $\{\mathcal{A}_F, \hat{U}_F/p, \mathcal{T}_F\}$ or by any pair (D_i, D_j) , $i \neq j$. This should be compared with [Va08, Theorem 2.7] which states (in our notations) that $D_i/D_i \cap D_j \simeq p^t \Delta_{i,j}$, where t is the maximal integer such that $i \equiv j \pmod{[F(\mu_{p^t}) : F]}$ and $\Delta_{i,j} \subset H^2(G_S(F), \mathbb{Z}_p(j))$ is the image, by corestriction, of $H_{Iw}^2(G_S(F_\infty), \mathbb{Z}_p(i))^\Gamma(j-i) \subset H_{Iw}^2(G_S(F_\infty), \mathbb{Z}_p(j))$. Here $H_{Iw}^2(G_S(F_\infty), \bullet)$ denotes, as usual, $\varprojlim H^2(G_S(F_n), \bullet)$ with respect to corestriction.

We already noticed (Remark (v) following Corollary 2.5) that \mathcal{A}_F and \hat{U}_F/p coincide when F contains enough p -primary roots of unity (see also Corollary 3.2 below). Proposition 2.7 shows that this is not the case in general, but their deviation, which is of an asymptotical nature, goes to zero when we go up the cyclotomic tower.

3 Upper bounds for the Kummer radical \mathcal{A}_F

We are going to give two "upper bounds" for \mathcal{A}_F . The first will be a "norm" radical which is accessible in the sense of the introduction. The second, via a local-global approach, will be a radical characterized by everywhere local embeddability in a \mathbb{Z}_p -extension.

3.1 Bounding from above by a norm radical

We keep the notations of the preceding sections.

Theorem 3.1. *Suppose that F contains μ_p and the layers F_n 's verify Gross' conjecture (i.e. $\Delta_\infty = 0$). Take m large enough for Γ_m to act trivially on X° and put $n = m + 1$. Then, for every $i \in \mathbb{Z}$, we have an exact sequence:*

$$0 \rightarrow \text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p] \rightarrow (\hat{U}'_n/p)^{G_n}(i) \rightarrow X^\circ(i)_\Gamma[p] \rightarrow 0.$$

In particular, if F also satisfies Leopoldt's conjecture, we have:

$$0 \rightarrow \mathcal{A}'_F \rightarrow (\hat{U}'_n/p)^{G_n} \rightarrow (X^\circ(-1))_\Gamma[p](1) \rightarrow 0$$

where $G_n = \text{Gal}(F_n/F)$.

Proof. Let us start from the exact sequence which appeared at the beginning of the proof of Theorem 2.1 (notice that by the hypothesis $\Delta_\infty = 0$, we have $\mathfrak{L} = \hat{\mathfrak{N}}$):

$$\begin{array}{ccccccc} 0 & \longrightarrow & X^\circ(i)^\Gamma & \longrightarrow & \tilde{\mathfrak{N}}(i)^\Gamma & \xrightarrow{\quad\quad\quad} & \hat{\mathfrak{N}}(i)^\Gamma \longrightarrow X^\circ(i)_\Gamma \longrightarrow 0 \\ & & & & \searrow & & \nearrow \\ & & & & \text{Div}(\hat{\mathfrak{N}}(i)^\Gamma) & & \end{array} \quad (2)$$

from which we derive by the snake lemma applied to the p -th power map:

$$0 \rightarrow \text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p] \rightarrow \hat{\mathfrak{N}}(i)^\Gamma[p] \rightarrow X^\circ(i)_\Gamma[p] \rightarrow 0.$$

It remains to give an adequate expression of $\hat{\mathfrak{N}}(i)^\Gamma[p]$. But $\hat{\mathfrak{N}}(i)^\Gamma[p] = \hat{\mathfrak{N}}^\Gamma[p](i)$ since F contains μ_p . Now, by Lemma 2.4, we have an exact sequence

$$0 \rightarrow \hat{U}'_n \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \hat{\mathfrak{N}}^{\Gamma_n} \rightarrow (X^\circ)_{\Gamma_n} \rightarrow 0,$$

for all n . Hence also

$$0 \rightarrow \hat{U}'_n/p \rightarrow \hat{\mathfrak{N}}^{\Gamma_n}[p] \rightarrow (X^\circ)_{\Gamma_n}[p] \rightarrow 0.$$

Take the fixed points by $G_n := \text{Gal}(F_n/F)$ and compare with the level of F . By Lemma 1.4, the natural map $\hat{U}'_F/p \rightarrow (\hat{U}'_n/p)^{G_n}$ is injective, hence we have a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\hat{U}'_n/p)^{G_n} & \longrightarrow & \hat{\mathfrak{N}}^\Gamma[p] & \longrightarrow & (X^\circ)_{\Gamma_n}[p]^{G_n} \\ & & \uparrow & & \parallel & & \uparrow \\ 0 & \longrightarrow & (\hat{U}'_F/p) & \longrightarrow & \hat{\mathfrak{N}}^\Gamma[p] & \longrightarrow & (X^\circ)_\Gamma[p] \longrightarrow 0. \end{array}$$

Let $\omega_n := \gamma^{p^n} - 1$. The vertical map on the right (which corresponds to restriction) is multiplication by ω_n/ω_0 . Choose m such that Γ_m acts trivially on X° , and take $n = m + 1$, so that $\frac{\omega_n}{\omega_m}$ annihilates $(X^\circ)_{\Gamma_m}[p] = X^\circ[p]$. Then $\frac{\omega_n}{\omega_0} = \frac{\omega_n}{\omega_m} \frac{\omega_m}{\omega_0}$ is the zero map on $(X^\circ)_\Gamma[p]$ and we also get an isomorphism $(\hat{U}'_n/p)^{G_n} \xrightarrow{\sim} \hat{\mathfrak{N}}^\Gamma[p]$. \square

The above Theorem gives a satisfactory description of the radical \mathcal{A}'_F in the terms set out in the introduction. The module $(\hat{U}'_n/p)^{G_n}$ is effectively accessible and the asymptotical deviation $(X^\circ(-1))_\Gamma[p](1)$ is bounded in the cyclotomic tower.

For completeness, let us now reprove along the same lines a slightly improved version of a result of Greenberg, generalized by Vauclair:

Corollary 3.2. ([Gre78, page 1242], [Va06, Theorem 2.2]) Let p^e be the exponent of X° . Assume that $\mu(F) = \mu_{p^a}$ with $a \geq e$, and that the conditions of Theorem 3.1 hold. Then $\mathcal{A}_F = \hat{U}_F/p$ if $a > e$. If $a = e$ and Γ acts trivially on X° , then $\mathcal{A}_F \neq \hat{U}_F/p$

Proof. According to the proof of Theorem 3.1, we have two exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{A}'_F & \rightarrow & \mathfrak{L}(-1)^\Gamma[p](1) = \mathfrak{L}^\Gamma[p] & \rightarrow & (X^\circ(-1))_\Gamma[p](1) \rightarrow 0 \\ & & & & \parallel & & \\ 0 & \rightarrow & \hat{U}'_F/p & \rightarrow & \hat{\mathfrak{N}}^\Gamma[p] & \rightarrow & (X^\circ)_\Gamma[p] \rightarrow 0 \end{array}$$

which we need to link by vertical maps becoming equalities. These exact sequences were obtained by Kummer duality from

$$\begin{array}{ccccccc} 0 & \rightarrow & H(1)^\Gamma/p & \xrightarrow{\sigma_1} & (\mathrm{fr}_\Lambda \mathfrak{X}_\infty(1))_\Gamma/p & \rightarrow & \mathrm{fr}_{\mathbb{Z}_p}((\mathrm{fr}_\Lambda \mathfrak{X}_\infty(1))_\Gamma)/p \rightarrow 0 \\ & & & & \parallel & & \\ 0 & \rightarrow & H^\Gamma/p & \xrightarrow{\sigma} & (\mathrm{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma/p & \rightarrow & \mathrm{fr}_{\mathbb{Z}_p}((\mathrm{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma)/p \rightarrow 0 \end{array}$$

where the maps σ and σ_1 originate from the snake lemma. More precisely the map $\sigma : H^\Gamma/p \rightarrow (\mathrm{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma/p = (\mathrm{fr}_\Lambda \mathfrak{X}_\infty)/(\omega, p)$ (where $\omega = \gamma - 1$) is defined from the exact sequence

$$0 \rightarrow \mathrm{fr}_\Lambda \mathfrak{X}_\infty \rightarrow \Lambda^{r_2} \rightarrow H \rightarrow 0$$

in the following way: for $h \in H^\Gamma$, let λ be any lift of h in Λ^{r_2} . Then $\omega(\lambda) \in \mathrm{fr}_\Lambda \mathfrak{X}_\infty$ and σ sends $h \bmod p$ to $\omega(\lambda) \bmod (\omega, p)$. Likewise, since $H^\Gamma = H(1)^\Gamma$ by hypothesis, we start with the same h that we lift to the same $\lambda \in \Lambda^{r_2}$. Put $\omega^{(1)} := \kappa(\gamma)\gamma - 1$. Then $\omega^{(1)}(\lambda) \in \mathrm{fr}_\Lambda \mathfrak{X}_\infty$ and σ_1 sends $h \bmod p$ to $\omega^{(1)}(\lambda) \bmod (\omega^{(1)}, p)$. Now, $(\omega^{(1)}, p) = (\omega, p)$ and $(\omega^{(1)} - \omega)(\lambda) \in p^{e+1}\Lambda^{r_2}$ by hypothesis. Therefore $(\omega^{(1)} - \omega)(\lambda) \in p \mathrm{fr}_\Lambda \mathfrak{X}_\infty$ since p^e annihilates H . We then have the following commutative square

$$\begin{array}{ccc} H(1)^\Gamma/p & \xrightarrow{\sigma_1} & (\mathrm{fr}_\Lambda \mathfrak{X}_\infty(1))_\Gamma/p \\ \parallel & & \parallel \\ H^\Gamma/p & \xrightarrow{\sigma} & (\mathrm{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma/p \end{array}$$

which implies the equality $\mathrm{fr}_{\mathbb{Z}_p}((\mathrm{fr}_\Lambda \mathfrak{X}_\infty(1))_\Gamma)/p = \mathrm{fr}_{\mathbb{Z}_p}((\mathrm{fr}_\Lambda \mathfrak{X}_\infty)_\Gamma)/p$, i.e. the equality $\mathcal{A}'_F = \hat{U}'_F/p$. Hence $\mathcal{A}_F = \hat{U}_F/p$.

Suppose now that $a = e$ and Γ acts trivially on X° . Then $H(1)^\Gamma = H^\Gamma = H$. Keeping the same notations, choose $h \in H$ of maximal order p^e . Then $(\omega^{(1)}, p) = (\omega, p)$ and $(\omega^{(1)} - \omega)(\lambda) \in p^e \Lambda^{r_2}$ and as before $(\omega^{(1)} - \omega)(\lambda) \in \mathrm{fr}_\Lambda \mathfrak{X}_\infty$. Putting $\kappa(\gamma) = 1 + up^e$, with $u \in \mathbb{Z}_p^*$, we have $x := (\omega^{(1)} - \omega)(\lambda) = up^e \gamma(\lambda) \in \mathrm{fr}_\Lambda \mathfrak{X}_\infty$. If $x \in p \mathrm{fr}_\Lambda \mathfrak{X}_\infty$, we would get $p^{e-1} \gamma(\lambda) \in \mathrm{fr}_\Lambda \mathfrak{X}_\infty$, contrary to the choice of h . We have thus shown that $\sigma_1(h) \neq \sigma(h)$ \square

Remark Of course, an analogous result holds when replacing the pair $(\mathcal{A}_F, \hat{U}_F/p)$ by any pair taken from $\{\mathcal{A}_F, \hat{U}_F/p, \mathcal{T}_F\}$ or by any pair (D_i, D_j) as in the remark following Proposition 2.7, satisfying Greenberg's conjecture alluded to in the proof of Proposition 2.3. For a general exponent p^k , see [Va06, Theorem 2.2].

3.2 Bounding from above by the Bertrandias-Payan radical

In this subsection, we introduce a certain Kummer radical \mathcal{B}_F (see [Ng86]) coming from global-local considerations concerning embeddability in cyclic p -extensions of arbitrary degrees. The radical \mathcal{B}_F contains all the previous ones $\mathcal{A}_F, \mathcal{T}_F$ and \hat{U}_F/p . The determination of the respective quotients sheds additional light on the interrelationship between the three radicals themselves.

Definition 3.3. A (necessarily cyclic) p -extension K/F is called *infinitely embeddable* (resp. \mathbb{Z}_p -embeddable) if it can be embedded in cyclic p -extensions of arbitrarily large degrees (resp. in a \mathbb{Z}_p -extension) of F .

The compositum of all the infinitely embeddable p -extensions of F is called the field of Bertrandias-Payan F^{BP} (in reference to [BP72]). An infinitely embeddable extension is necessarily unramified outside p -adic primes: $F^{BP} \subseteq \hat{F}$. Hence the Galois group $BP_F := \text{Gal}(F^{BP}/F)$ is a quotient of \mathfrak{X}_F . Moreover F^{BP} obviously contains the composite of all \mathbb{Z}_p -extensions of F so that

$$\text{fr}_{\mathbb{Z}_p} BP_F = \text{fr}_{\mathbb{Z}_p} \mathfrak{X}_F.$$

The following criterion is a consequence of class field theory (see [BP72]):

Proposition 3.4. Assume that F contains $\mu_p = \langle \zeta_p \rangle$ and let $K := F(\sqrt[p]{a})$, $a \in F^\bullet$ be a cyclic extension of degree p . The following conditions are equivalent:

- (i) $F(\sqrt[p]{a})/F$ is infinitely embeddable;
- (ii) $F_v(\sqrt[p]{a})/F_v$ is \mathbb{Z}_p -embeddable for all finite primes v ;
- (iii) $a \in F_v^{\bullet p} \hat{F}_v^\bullet$ for all finite primes v (note that $\hat{F}_v^\bullet = U_v$ for $v \nmid p$);
- (iv) $(a, \zeta_p)_v = 1$ for all finite primes v . Here $(\cdot, \cdot)_v$ stands for the maximal degree Hilbert symbol attached to the local field F_v ;
- (v) the symbol $\{a, \zeta_p\}$ belongs to the wild kernel $WK_2(F)$, i.e. the intersection in $K_2(F)$ of the kernels of all the Hilbert symbols.

Definition 3.5. The radical of Bertrandias-Payan \mathcal{B}_F is, by definition, the subgroup of $F^\bullet/F^{\bullet p}$ consisting of classes $a \bmod F^{\bullet p}$ such that a verifies the preceding equivalent conditions. Clearly \mathcal{B}_F contains $\mathcal{A}_F, \hat{U}_F/p$ and the Tate kernel \mathcal{T}_F . Condition (i) means that $\mathcal{B}_F = \text{Hom}(BP_F, \mu_p)$ and the quotient $\mathcal{B}_F/\mathcal{A}_F$ measures the obstruction between "global \mathbb{Z}_p -embeddability" and "everywhere local \mathbb{Z}_p -embeddability". Condition (iii) (resp. (iv), resp. (v)) says that \mathcal{B}_F coincides with the radical denoted by $D_F^{(1)}$ (resp. $B/F^{\bullet p}$, resp. $\mathcal{E}/F^{\bullet p}$) in [Ko91] (resp. [Gre78], resp. [KC78, section 3]).

Theorem 3.6. Suppose that F contains μ_p . Then

- (i) $\mathcal{B}_F/\mathcal{T}_F \simeq X_\infty(1)_\Gamma[p](-1)$
- (ii) $\mathcal{B}_F/\mathcal{A}_F \simeq X_\infty(-1)_\Gamma[p](1)$
- (iii) $\mathcal{B}_F/(\hat{U}_F/p) \simeq (X_\infty)_\Gamma[p]$.

Proof. Tate's theorem [Ta76] asserts the existence of an exact sequence

$$0 \rightarrow \mathcal{T}_F(1) \rightarrow F^\bullet/F^{\bullet p}(1) \xrightarrow{\alpha} K_2(F)[p] \rightarrow 0,$$

where the map α is defined by $\alpha(a \otimes \zeta_p) = \{a, \zeta_p\}$. Put $H^1(G_S(F), \mu_p) = \Delta_F/F^{\bullet p}$, where $\Delta_F := \{a \in F^\bullet / v(a) \equiv 0 \pmod{p} \forall v \nmid p\}$. Then, if we restrict the map α to $\Delta_F/F^{\bullet p}(1)$, the above exact sequence becomes

$$0 \rightarrow \mathcal{T}_F(1) \rightarrow \Delta_F/F^{\bullet p}(1) \xrightarrow{\alpha} R_2(F)[p] \rightarrow 0,$$

where $R_2(F)$ denotes the tame kernel of F , i.e. the intersection in $K_2(F)$ of the kernels of all the tame symbols. By property (v) of Proposition 3.4, \mathcal{B}_F is the inverse image of $WK_2(F)[p]$ under the map α , hence an exact sequence

$$0 \rightarrow \mathcal{T}_F(1) \rightarrow \mathcal{B}_F(1) \xrightarrow{\alpha} WK_2(F)[p] \rightarrow 0.$$

But it is known that the p -primary part of the wild kernel $WK_2(F)\{p\}$ can be expressed as a codescent module: the exact sequence

$$0 \rightarrow WK_2(F)\{p\} \rightarrow H^2(G_S(F), \mathbb{Z}_p(2)) \rightarrow \oplus_{v|p} H^2(F_v, \mathbb{Z}_p(2))$$

together with Poitou-Tate duality imply that

$$WK_2(F)\{p\}^* \simeq \text{Ker}_S^1(F, \mathbb{Q}_p/\mathbb{Z}_p(-1)) := \text{Ker}(H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(-1)) \rightarrow \oplus_{v|p} H^1(F_v, \mathbb{Q}_p/\mathbb{Z}_p(-1))).$$

The cohomological triviality of $\mathbb{Q}_p/\mathbb{Z}_p(-1)$ (Tate's Lemma) then shows that

$$\text{Ker}_S^1(F, \mathbb{Q}_p/\mathbb{Z}_p(-1))^* \simeq X_\infty(1)_\Gamma.$$

Thus (i) is proved.

(ii) The exact sequence

$$0 \rightarrow \operatorname{tor}_{\mathbb{Z}_p} BP_F \rightarrow BP_F \rightarrow \operatorname{fr}_{\mathbb{Z}_p} BP_F = \operatorname{fr}_{\mathbb{Z}_p} \mathfrak{X}_F \rightarrow 0,$$

yields, by Kummer duality, an exact sequence

$$0 \rightarrow \mathcal{A}_F \rightarrow \mathcal{B}_F \rightarrow \operatorname{Hom}(\operatorname{tor}_{\mathbb{Z}_p} BP_F, \mu_p) \rightarrow 0.$$

It remains to determine $\operatorname{Hom}(\operatorname{tor}_{\mathbb{Z}_p} BP_F, \mu_p) \simeq (\operatorname{tor}_{\mathbb{Z}_p} BP_F/p)^*(1) \simeq (\operatorname{tor}_{\mathbb{Z}_p} BP_F)^*[p](1)$. We will follow the proof of Theorem 4.2 of [Ng86] (but without appealing to Leopoldt's conjecture). The equivalence between the first two parts of Proposition 3.4 shows that

$$(BP_F)^* = \{\chi \in \mathfrak{X}_F^* / \chi_v \in \operatorname{Div}(\mathfrak{X}_v^*), \forall v \mid p\}.$$

Here \mathfrak{X}_v is the Galois group over F_v of the maximal abelian pro- p -extension of F_v and χ_v is the character obtained by restricting χ to \mathfrak{X}_v . Then

$$\begin{aligned} (\operatorname{tor}_{\mathbb{Z}_p} BP_F)^* &\simeq (BP_F)^* / \operatorname{Div}(\mathfrak{X}_F^*) \\ &\simeq \operatorname{Ker}(\mathfrak{X}_F^* / \operatorname{Div} \rightarrow \oplus_{v \mid p} \mathfrak{X}_v^* / \operatorname{Div}) \\ &\simeq \operatorname{Ker}(H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p) / \operatorname{Div} \rightarrow \oplus_{v \mid p} H^1(F_v, \mathbb{Q}_p/\mathbb{Z}_p) / \operatorname{Div}). \end{aligned}$$

By a result of Tate in p -adic cohomology [Ta76, section 2], the boundary map induces an isomorphism $H^1(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p) / \operatorname{Div} \simeq \operatorname{tor}_{\mathbb{Z}_p} H^2(G_S(F), \mathbb{Z}_p)$ (and similarly for the local cohomology groups), hence

$$(\operatorname{tor}_{\mathbb{Z}_p} BP_F)^*[p] \simeq \operatorname{Ker}(H^2(G_S(F), \mathbb{Z}_p) \rightarrow \oplus_{v \mid p} H^2(F_v, \mathbb{Z}_p))[p].$$

By Poitou-Tate's duality, $\operatorname{Ker}(H^2(G_S(F), \mathbb{Z}_p) \rightarrow \oplus_{v \mid p} H^2(F_v, \mathbb{Z}_p))$ is dual to $\operatorname{Ker}_S^1(F, \mathbb{Q}_p/\mathbb{Z}_p(1))$ (the notation is analogous to that of (i)), and Tate's lemma once again shows that $\operatorname{Ker}_S^1(F, \mathbb{Q}_p/\mathbb{Z}_p(1))^* \simeq X_\infty(-1)_\Gamma$. Thus (ii) is proved.

(iii) Coming back to Sinnott's exact sequence, let us note that X_∞ is actually the Galois group over F_∞ of the maximal abelian pro- p -extension of F_∞ which is totally decomposed at every finite place of F_∞ . Hence, for any set of places T containing the p -adic primes, we have the T -analogue of Sinnott's exact sequence

$$\bar{U}_F^T \longrightarrow \bigoplus_{v \in T} \bar{F}_v^\bullet / \tilde{F}_v^\bullet \xrightarrow{\operatorname{Artin}} (X_\infty)_\Gamma \rightarrow A_F^T \rightarrow 0,$$

where \bar{U}_F^T denotes the pro- p -completion of the group of T -units of F and A_F^T the p -group of T -classes of F . Taking T to be the set of all finite primes of F , we get an exact sequence

$$\bar{F}^\bullet \xrightarrow{g} \bigoplus_{v \nmid \infty} \bar{F}_v^\bullet / \tilde{F}_v^\bullet \xrightarrow{\operatorname{Artin}} (X_\infty)_\Gamma \rightarrow 0,$$

where \bar{F}^\bullet denotes the pro- p -completion of F^\bullet , which injects into $\bigoplus_{v \nmid \infty} \bar{F}_v^\bullet$ by Hasse's principle. The kernel of g consists of elements $x \in \bar{F}^\bullet$ such that $x \in \tilde{F}_v^\bullet$ for all finite primes v . Since $\tilde{F}_v^\bullet = U_v$ for any $v \nmid p$, it follows at once that $\operatorname{Ker} g = \hat{U}_F$. Applying the snake lemma for the multiplication by p , we readily get a commutative diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & (X_\infty)_\Gamma[p] & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \hat{U}_F'/p & \longrightarrow & F^\bullet / F^{\bullet p} & \longrightarrow & \operatorname{im} g/p \longrightarrow 0 \\ & & & & \searrow & & \downarrow \\ & & & & & & \bigoplus_{v \nmid \infty} \bar{F}_v^\bullet / \tilde{F}_v^\bullet F_v^{\bullet p} \end{array}$$

By property (iii) of Proposition 3.4, the kernel of the dotted arrow is simply the radical of Bertrandias-Payan \mathcal{B}_F and a diagram chasing immediately leads to:

$$\mathcal{B}_F/(\hat{U}_F/p) \simeq (X_\infty)_\Gamma[p].$$

Thus (iii) is proved. \square

Remarks: Idelic proofs of properties (i) and (iii) can be found in [Ko91, pages 18 and 14]. Also, compare (ii) with the main result of [Se11].

4 Study of a particular case

Although essentially our approach has a theoretical orientation, it lends itself to effective or algorithmic calculations, as will be shown in the examples below.

4.1 The case where X° is of order p

We continue to assume the two standard conjectures : Leopoldt's for our base field F and Gross' for all the layers F_n . As noticed before (proof of Theorem 3.1), the short exact sequence at the infinite level

$$0 \rightarrow X^\circ(i) \rightarrow \tilde{\mathfrak{N}}(i) \rightarrow \hat{\mathfrak{N}}(i) \rightarrow 0$$

yields

$$\begin{array}{ccccccc} 0 & \longrightarrow & X^\circ(i) & \longrightarrow & \tilde{\mathfrak{N}}(i)^\Gamma & \xrightarrow{f_i} & \hat{\mathfrak{N}}(i)^\Gamma \longrightarrow X^\circ(i) \longrightarrow 0 \\ & & & & \searrow & & \swarrow \\ & & & & \text{Div } \hat{\mathfrak{N}}(i)^\Gamma & & \end{array} \quad (2)$$

which gives by applying the snake lemma to multiplication-by- p and assuming that X° is of order p

$$0 \rightarrow X^\circ(i) \rightarrow \tilde{\mathfrak{N}}(i)^\Gamma[p] = \tilde{\mathfrak{N}}^\Gamma[p](i) \rightarrow \text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p] \rightarrow X^\circ(i) \rightarrow 0.$$

We are interested in $\text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p]$ for which the above exact sequence only provides a hyperplane. But in the exact sequence (2), the surjectivity of $\tilde{\mathfrak{N}}(i)^\Gamma \rightarrow \text{Div } \hat{\mathfrak{N}}(i)^\Gamma$ shows that each $\beta_i \in \text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p]$ has a pre-image $\tilde{\beta}_i \in \tilde{\mathfrak{N}}(i)^\Gamma[p^2]$. In fact, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X^\circ(i) & \longrightarrow & \tilde{\mathfrak{N}}(i)^\Gamma[p] & \longrightarrow & \text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p] \longrightarrow X^\circ(i) \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & X^\circ(i) & \longrightarrow & f_i^{-1}(\text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p]) & \xrightarrow{f_i} & \text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p] \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X^\circ(i) & \longrightarrow & \tilde{\mathfrak{N}}(i)^\Gamma[p^2] & \xrightarrow{g_i} & \text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p^2] \longrightarrow X^\circ(i) \longrightarrow 0 \end{array} \quad (3)$$

and we are going to devise an algorithm determining $\mathcal{D}_i := f_i^{-1}(\text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p])$.

To this end, we notice first that

$$\tilde{\mathfrak{N}}(i)^\Gamma[p^2] = (\tilde{\mathfrak{N}}(i)[p^2]^{\Gamma_1})^{G_1} = (\tilde{\mathfrak{N}}^{\Gamma_1}[p^2](i))^{G_1} \simeq (\tilde{U}'_1/p^2)(i)^{G_1} \quad (\text{Lemma 2.2})$$

where $\Gamma_1 := \text{Gal}(F_\infty/F_1)$ and $G_1 := \text{Gal}(F_1/F)$. Since $\tilde{U}'_1/p^2 \simeq (\bar{U}'_\infty)_{\Gamma_1}/p^2$ is free as a module over $\mathbb{Z}/p^2[G_1]$ (Proposition 1.1), the fixed points $(\tilde{U}'_1/p^2)(i)^{G_1}$ can be identified with $\nu^{(i)}(\tilde{U}'_1/p^2)$, where ν is the norm (trace in additive notation) relative to G_1 and $\nu^{(i)}$ is its i -twist. The above diagram (3) then shows that \mathcal{D}_i consists of all the $\nu^{(i)}(\bar{x})$, $\bar{x} \in \tilde{U}'_1/p^2$ such that $p(g_i(\nu^{(i)}(\bar{x}))) = 0$, or, according to the third line of (3), such that $p(\nu^{(i)}(\bar{x})) \in X^\circ(i)$. Thus, we are left to express $X^\circ(i)$ in this

context. We suppose that \hat{U}'_F is known (by Lemma 1.4) hence also \tilde{U}'_F by Proposition 1.5. In fact, since they are both free \mathbb{Z}_p -modules with $[\hat{U}'_F : \tilde{U}'_F] = p$, there exists a \mathbb{Z}_p -basis (e_1, e_2, \dots) of \hat{U}'_F such that (pe_1, e_2, \dots) is a \mathbb{Z}_p -basis of \tilde{U}'_F . The first line of (3) shows immediately that $X^\circ(i)$ is the cyclic group generated by the class (in \tilde{U}'_F/p) of pe_1 , which can be written as $pe_1 \otimes 1/p \in \hat{\mathfrak{N}}(i)^\Gamma[p]$.

Now, writing $\bar{x} \in \tilde{U}'_1/p^2$ as $x \otimes 1/p^2 \in \hat{\mathfrak{N}}(i)^\Gamma[p^2]$, with $x \in \tilde{U}'_1$, we conclude our algorithm:

Proposition 4.1. *The pre-image $\mathcal{D}_i = f_i^{-1}(\text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[p])$ consists of the elements $\nu^{(i)}(x) \otimes 1/p^2 \in \hat{\mathfrak{N}}(i)^\Gamma[p^2]$, $x \in \tilde{U}'_1$ such that $\nu^{(i)}(x) \otimes 1/p$ is colinear to $pe_1 \otimes 1/p$.*

Remarks: (i) The colinearity factor above is in \mathbb{Z}/p . It is not zero precisely when $\nu^{(i)}(x) \otimes 1/p^2$ is of order p^2 .

(ii) For practical purposes, it is important to stress that the maps f_i and g_i in diagram (3) are induced by the natural map $\tilde{\mathfrak{N}} \rightarrow \hat{\mathfrak{N}}$.

(iii) Our algorithm can be extended to the general case (i.e. when X° is not necessarily of order p), but the calculations of course become heavier, because F_1 must be replaced by some higher level F_n , and the preliminary determination of X° (considered for example as a capitulation kernel; see the remark at the end of Section 2) is not a trivial matter.

(iv) As in the already existing effective ([Gra85, He88]) or algorithmic ([Th93]) calculations of \mathcal{A}_F , we need information on the p -units (via \hat{U}'_F) and the p -class groups (via X°). But note that the methods based on class field theory (op. cit.) need further to appeal to explicit reciprocity laws (via Hilbert symbols).

4.2 Biquadratic fields for $p = 3$

The biquadratic fields $F := \mathbb{Q}(\mu_3, \sqrt{d})$ for $p = 3$ with $d \in \mathbb{Z}$ squarefree and $3 \nmid d$ were first studied by Kramer and Candiotti [KC78] who determined \mathcal{A}_F for $|d| < 200$ and showed that $\mathcal{A}_F = \mathcal{T}_F$, except for the five critical values $d = -107, 67, 103, 106$ and 139 which they did not treat. Immediately after, Greenberg [Gre78] showed that $\mathcal{A}_F \neq \mathcal{T}_F$ for $d = 67$. The corresponding \mathcal{A}_F for these critical cases were computed by H  mard [He88] using an idelic calculation of the \mathbb{Z}_p -torsion of the Bertrandias-Payan module BP_F (see Section 3.2). A similar approach allowed Thomas [Th93] to devise an algorithm computing \mathcal{A}_F for a wide range of d . We do not intend to redo these calculations. It seems more interesting to use our approach to determine the Tate kernel \mathcal{T}_F because, curiously enough, we know of no such systematic computation in the literature. For $p = 3$, there are only three relevant kernels $\hat{U}_F/3$, \mathcal{A}_F and \mathcal{T}_F corresponding to the twists $0, -1$ and 1 respectively. For the biquadratic fields F , we aim to derive the Tate kernel from the knowledge of the two other kernels.

Let us consider the five critical cases $F := \mathbb{Q}(\mu_3, \sqrt{d})$, $d = -107, 67, 103, 106$ and 139 with $p = 3$. Using PARI package, we can see that $X^\circ(F^+)$ is of order 3, hence the algorithm of Section 4.1 applies. It is easy to check that $F(\sqrt[3]{3})$ is the first layer of the anticyclotomic \mathbb{Z}_3 -extension of F and 3 is a global universal norm. The three distinct kernels $\hat{U}'_F/3$, \mathcal{A}'_F and \mathcal{T}'_F have dimension 2 and contain $\bar{3} := 3 \bmod F^{\bullet 3}$. Let us treat in detail a specific example, say $d = 67$. The fundamental unit of the quadratic subfield $F^+ = \mathbb{Q}(\sqrt{67})$ is $\epsilon = 48842 + 5967\sqrt{67}$ and Lemma 1.4 shows that $\bar{\epsilon} \in \hat{U}'_F/3$ (but $\epsilon \notin \hat{U}'_F$). Then $\hat{U}'_F/3 = \langle \bar{3}, \bar{\epsilon} \rangle$ whereas $\mathcal{A}'_F = \langle \bar{3}, \bar{\epsilon}^2 \bar{\eta} \rangle$ ([He88, page 371]) with $\eta = 8 + \sqrt{67}$ being a generator of a 3-adic prime in F^+ . These radicals are hyperplanes of the $\mathbb{Z}/3$ -vector space $\hat{U}'_F/3 = \langle \bar{3}, \bar{\epsilon}, \bar{\eta} \rangle$, but also of the $\mathbb{Z}/3$ -vector space $(\hat{U}'_1/3)^{G_1}$ according to Theorem 3.1. Their explicit form shows that a basis of $(\hat{U}'_1/3)^{G_1}$ will be $\{\bar{3}, \bar{\epsilon}, \bar{\epsilon}^2 \bar{\eta}\}$, or better $\{\bar{3}, \bar{\epsilon}, \bar{\eta}\}$. In particular, $(\hat{U}'_1/3)^{G_1}$ coincides with the image of the natural injection $\hat{U}'_F/3 \hookrightarrow \hat{U}'_1/3$. Obviously the third kernel \mathcal{T}'_F will be of the form $\{\bar{3}, \bar{\eta}\}$ or $\{\bar{3}, \bar{\eta} \bar{\epsilon}\}$. But the choice between these two forms will unexpectedly be non trivial.

Since the rest of the calculation will essentially be elementary linear algebra, it will be more convenient to use additive notation for the three kernels $D_i := \text{Div}(\hat{\mathfrak{N}}(i)^\Gamma)[3]$. They could be put on the same level and computed using the algorithm of Section 4.1, but here we want to deduce the last one from the two others. Let us first fix some general notations: $D_i = \langle 3 \otimes \frac{1}{3}, \beta_i \rangle$, $\beta_0 = \epsilon \otimes \frac{1}{3}$,

$\beta_{-1} = (\eta - \epsilon) \otimes \frac{1}{3}$. According to diagram (3) of Section 4.1, $\mathcal{D}_i = f_i^{-1}(D_i)$ is of type $(3^2, 3)$ and $f_i(\mathcal{D}_i[3]) = f_i(\tilde{\mathfrak{N}}(i)^\Gamma[3])$ is the cyclic group generated by $3 \otimes \frac{1}{3}$. We look for pre-images (necessarily of order 3^2) $\tilde{\beta}_i \in \mathcal{D}_i$ of β_i such that $\mathcal{D}_i = \langle 3 \otimes \frac{1}{3}, \tilde{\beta}_i \rangle$.

Recall that Proposition 4.1 gives us the general form of the $\tilde{\beta}_i$'s, but here we directly know (by our chosen approach) β_0 and β_{-1} . Let us fix $\epsilon' \in \hat{U}'_F$ such that $\epsilon' \otimes \frac{1}{3} = \epsilon \otimes \frac{1}{3}$ in $\hat{\mathfrak{N}}$. We have just seen that $f_0(\tilde{\mathfrak{N}}^\Gamma[3]) = \langle 3 \otimes \frac{1}{3} \rangle$, hence $\epsilon' \otimes \frac{1}{3} \notin \tilde{\mathfrak{N}}^\Gamma[3]$ or equivalently $\epsilon' \notin \tilde{U}'_F$. Because $3\epsilon' \in \tilde{U}'_F$, a pre-image $\tilde{\beta}_0$ of β_0 will be $3\epsilon' \otimes \frac{1}{3^2}$. Similarly, consider $\eta \otimes \frac{1}{3} \in (\hat{U}'_1/3)^{G_1}$ and fix $\eta' \in \hat{U}'_1$ such that $\eta' \otimes \frac{1}{3} = \eta \otimes \frac{1}{3}$. We have seen that $(\tilde{U}'_1/3)^{G_1} = \nu(\tilde{U}'_1/3)$ and therefore $(\tilde{U}'_1/3)^{G_1} \simeq \tilde{U}'_F/3$ since the (arithmetic) norm of $\tilde{U}'_1/3$ is $\tilde{U}'_F/3$. The same argument as for ϵ' allows us to show that $\eta' \notin \tilde{U}'_1$, and the natural map $\tilde{\mathfrak{N}}[3^2] \rightarrow \hat{\mathfrak{N}}[3^2]$ (whose kernel is X°) sends $3(\eta' - \epsilon') \otimes \frac{1}{3^2}$ to $(\eta - \epsilon) \otimes \frac{1}{3}$, but be aware that $3(\eta' - \epsilon') \otimes \frac{1}{3^2}$ is not a priori in $\tilde{\mathfrak{N}}(-1)^\Gamma[3^2]$. Actually, Proposition 4.1 shows that there exists an element $\nu^{(-1)}(x) \otimes \frac{1}{3^2} \in \mathcal{D}_{-1}$ such that $g_{-1}(\nu^{(-1)}(x) \otimes \frac{1}{3^2}) = (\eta - \epsilon) \otimes \frac{1}{3^2}$, hence $\nu^{(-1)}(x) \otimes \frac{1}{3^2}$ and $3(\eta' - \epsilon') \otimes \frac{1}{3^2}$ differ by an element of order 3 in $\tilde{\mathfrak{N}}[3^2]$. Therefore, we can take $\tilde{\beta}_{-1} = 3(\eta' - \epsilon') \otimes \frac{1}{3^2} + \delta_{-1}$ with $\delta_{-1} \in \tilde{\mathfrak{N}}[3]$. The additional condition in Proposition 4.1 then reads: $3(\eta' - \epsilon') \otimes \frac{1}{3} = \pm 3\epsilon' \otimes \frac{1}{3}$. The sign -1 on the right hand side would mean that $3\eta' \otimes \frac{1}{3} = 0$ in $\tilde{U}'_1/3$, i.e. $\eta' \in \tilde{U}'_1$ since \tilde{U}'_1 is torsion free: a contradiction. Finally, $\tilde{\beta}_{-1} = 3(\eta' - \epsilon') \otimes \frac{1}{3^2} + \delta_{-1}$ and $3\tilde{\beta}_{-1} = 3\epsilon' \otimes \frac{1}{3}$.

We must now give a general expression for the elements of order 3^2 in \mathcal{D}_1 . If $\tilde{\beta}_1$ is such an element, any other is obtained by adding an element of order 3 to $\pm\tilde{\beta}_1$. So we start by constructing a particular $\tilde{\beta}_1$. Fix $x_0, x_{-1} \in \tilde{U}'_1$ such that $\nu(x_0) \otimes \frac{1}{3} = \tilde{\beta}_0$ and $\nu^{(-1)}(x_{-1}) \otimes \frac{1}{3} = \tilde{\beta}_{-1}$ (see Proposition 4.1), and put $x_1 = x_0 + \lambda x_{-1}$, $\lambda = \pm 1$. An elementary calculation on Tate twists shows that $\nu^{(1)}(x_i)$ differs from $\nu^{(i)}(x_i)$ by an element of $(\tilde{U}'_1)^3$, hence $\tilde{\beta}_1 := \nu^{(1)}(x_1) \otimes \frac{1}{3} = \tilde{\beta}_0 + \lambda\tilde{\beta}_{-1} + \delta_1$, with $\delta_1 \in \tilde{\mathfrak{N}}[3]$. The additional condition in Proposition 4.1 reads: $3\tilde{\beta}_1 = 3\tilde{\beta}_0 + \lambda 3\tilde{\beta}_{-1} = \pm(3\epsilon' \otimes \frac{1}{3})$. Since $3\tilde{\beta}_0 = 3\epsilon' \otimes \frac{1}{3} = 3\tilde{\beta}_{-1}$ (see the previous calculations), we get $1 + \lambda = \pm 1$. The only possibility is $\lambda = 1$. Hence $\tilde{\beta}_1 = \tilde{\beta}_0 + \tilde{\beta}_{-1} + \delta_1$ and $f_1(\tilde{\beta}_1) = \eta \otimes \frac{1}{3} + f_1(\delta_1)$, so that any element of D_1 will be of the form $\pm\eta \otimes \frac{1}{3} + \delta'_1$, where δ'_1 is in the image of the natural map $\tilde{\mathfrak{N}}[3] \rightarrow \hat{\mathfrak{N}}[3]$. We can now choose between the two possibilities $\eta \otimes \frac{1}{3}$ or $(\eta + \epsilon) \otimes \frac{1}{3}$. In the second case, we would have $\epsilon \otimes \frac{1}{3}$ or $(\epsilon - \eta) \otimes \frac{1}{3} = \delta'_1$, whence $\tilde{\beta}_0$ or $\tilde{\beta}_{-1}$ would be of order 3 (because the kernel X° is of order 3): a contradiction. In conclusion, the Tate kernel in our example is $\mathcal{T}'_F = \langle \bar{3}, \bar{\eta} \rangle$: the symbol $\{\zeta_3, \eta\}$ is trivial in $K_2(F)$ whereas $\{\zeta_3, \epsilon\}$ is a non-trivial element of the wild kernel $WK_2(F)$.

Remark: The above result could of course be reached by describing the whole tame kernel $K_{2\mathcal{O}_F}$ by generators and relations. Algorithms for such a calculation exist in the literature [BG04]. However, Karim Belabas kindly pointed out to us that in the case of the above example, the generators are S -units, where S contains all the primes of norm less than 5096521 so by brute force calculation it would take years to find the relations between them.

References

- [AM04] **J. Assim, A. Movahhedi** *Bounds for étale capitulation kernels.* *K-Theory* **33** (2004), no. 3, 199–213.
- [BG04] **K. Belabas and H. Gangl** *Generators and relations for K_2O_F .* *K-Theory* **31** (2004), no. 3, 195–231.
- [BP72] **F. Bertrandias et J.-J. Payan** *Γ -extensions et invariants cyclotomiques.* *Ann. Sci. Éc. Norm. Sup.* **5** (1972), 517–543.
- [CK76] **J. Carroll and H. Kisilevsky** *Initial layers of \mathbb{Z}_ℓ -extensions of complex quadratic fields.* *Compositio Math.* (2)**32** (1976), 157–168.
- [Co72] **J. Coates** *On K_2 and some classical conjectures in algebraic number theory.* *Ann. of Math.* (2)**95** (1972), 99 – 116.
- [Co73] **J. Coates** *Research problems: arithmetic questions in K-theory.* Algebraic K-theory II, Battelle Institute Conference, Lecture Notes in Math. **341** (1973), pp. 521–523.
- [FGS81] **J. L. Federer and B. H. Gross (with an appendix by W. Sinnott)** *Regulators and Iwasawa modules.* *Invent. Math.* **62** (1981), 443–457.
- [FN91] **V. Fleckinger, T. Nguyen Quang Do** *Bases normales, unités et conjecture faible de Leopoldt.* *Manuscripta Math.* **71** (1991), no. 2, 183–195.
- [Gra85] **G. Gras** *Plongements kummériens dans les \mathbb{Z}_p -extensions.* *Compositio Math.* **55** (1985), no. 3, 383–396.
- [Gre73] **R. Greenberg** *On a certain l -adic representation.* *Invent. Math.* **21** (1973), no. 6, 117–124.
- [Gre78] **R. Greenberg** *A note on K_2 and the theory of \mathbb{Z}_p -extensions.* *Amer. J. Math.* **100** (1978), no. 6, 1235–1245.
- [Grei94] **C. Greither** *Sur les normes universelles dans les \mathbb{Z}_p -extensions.* *J. Théor. Nombres Bordeaux* **6** (1994), no. 2, 205–220.
- [He88] **D. Hémard** *Modules galoisiens de torsion et plongements dans les \mathbb{Z}_p -extensions.* *J. Number Theory* **30** (1988), no. 3, 357–374.
- [Hu05] **K. Hutchinson** *Tate Kernels, Étale K-Theory and The Gross Kernel.* Preprint (2005).
- [Iw73] **K. Iwasawa** *On \mathbb{Z}_ℓ -extensions of algebraic number fields.* *Ann. of Math.* (2) **98** (1973), 246–326.
- [Ja87] **J.-F. Jaulent** *Sur les conjectures de Leopoldt et de Gross.* *J. Arithm. de Besançon* (1985), Astérisque, **147-148** (1987), 107–120.
- [Ka06] **K. Kato**, *Universal norms of p -units in some non-commutative Galois extensions.* *Doc. Math.* (2006), Extra Vol., 551–565.
- [Ko91] **M. Kolster**, *An idelic approach to the wild kernel.* *Invent. Math.* **103** (1991), no. 1, 9–24.
- [KM00] **M. Kolster, A. Movahhedi**, *Galois co-descent for étale wild kernels and capitulation,* *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 1, 35–65.
- [KNF96] **M. Kolster, T. Nguyen Quang Do, V. Fleckinger** *Twisted S -units, p -adic class number formulas, and the Lichtenbaum conjectures.* *Duke Math. J.* **84** (1996), no. 3, 679–717.
- [KC78] **Kramer, Kenneth; Candiotti, Alan** *On K_2 and \mathbb{Z}_l -extensions of number fields.* *Amer. J. Math.*, **100**(1978), no. 1, 177–196.
- [Ku72] **L. V. Kuz'min**, *The Tate module for algebraic number fields,* *Math. USSR Izv* **6-2** (1972), no. 1, 263–321.
- [MS03] **McCallum, William G.; Sharifi, Romyar T.** *A cup product in the Galois cohomology of number fields.* *Duke Math. J.* **120** (2003), no. 2, 269–310.
- [Ng86] **T. Nguyen Quang Do** *Sur la \mathbb{Z}_p -torsion de certains modules galoisiens.* *Ann. Inst. Fourier (Grenoble)* **36** (1986), no. 2, 27–46.
- [Ng88] **T. Nguyen Quang Do** *Sur la torsion de certains modules galoisiens. II.* Séminaire de Théorie des nombres, Paris 1986–87, 271–297, *Progr. Math.*, 75, Birkhäuser Boston, MA, 1988.

- [Sc79] **P. Schneider**, *Über gewisse Galoiscohomologiegruppen*. Math. Z. **168** (1979), no. 2, 181–205.
- [Se11] **S. Seo**, *On first layers of \mathbb{Z}_p -extensions II* Acta Arith. **150** (2011), no. 4, 385–397.
- [Ta73] **J. Tate**, *Letter from Tate to Iwasawa on a relation between K_2 and Galois cohomology*. Algebraic K -theory, II: "Classical" algebraic K -theory and connections with arithmetic (Proc. Conf., Seattle Res. Center, Battelle Memorial Inst., 1972), Lecture Notes in Math. **342**, Springer, Berlin, (1973) 524–527.
- [Ta76] **J. Tate**, *Relations between K_2 and Galois cohomology*. Invent. Math. **36** (1976), 257–274.
- [Th93] **H. Thomas**, *Étage initial d'une \mathbb{Z}_ℓ -extension*. Manuscripta Math. **81** (1993), no 3–4, 413–435.
- [Va06] **D. Vauclair** *Sur la comparaison des noyaux de Tate d'ordre supérieur*. Algèbre et théorie des nombres. Années 2003–2006, 117–127, Publ. Math. Univ. Franche-Comté, Besançon 2006.
- [Va08] **D. Vauclair** *Noyaux de Tate et capitulation*. J. Number Theory **128** (2008), no. 3, 619–638.
- [Wa97] **Lawrence C. Washington** *Introduction to cyclotomic fields*. Second edition. Graduate Texts in Mathematics **83**. Springer-Verlag, New York, 1997.

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